PHYSICS 115B
HOMEWORK 9

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1 (a) look at expectation value

$$\langle x\rangle = \int_{-\infty}^{\infty} dx\, \psi^*(x)\, \hat{x}\, \psi(x) = \int_{-\infty}^{\infty} dx\, \psi^*(x)\, x\, \psi(x)$$

After a parity transformation

$$\langle x\rangle \longrightarrow \int_{-\infty}^{\infty} dx\, \psi^*(-x)\, x\, \psi(-x)$$

write $y=-x$. Change of variable!

$$\int_{-\infty}^{\infty} \rightarrow \int_{+\infty}^{-\infty} \qquad x \rightarrow -y \qquad dx = -dy$$

$$\langle x\rangle \longrightarrow \int_{\infty}^{-\infty} (-dy)\, \psi^*(y)\, (-y)\, \psi(y)$$

$$\langle x\rangle \longrightarrow \int_{\infty}^{-\infty} dy\, \psi^*(y)\, y\, \psi(y) = -\int_{-\infty}^{\infty} dx\, \psi^*(x)\, x\, \psi(x)$$

$$\langle x\rangle \rightarrow -\langle x\rangle$$

We could have also written

$$\langle x\rangle \rightarrow \int_{-\infty}^{\infty} \psi^*(x)\left[\Pi^\dagger \hat{x}\, \Pi\right] \psi(x)$$

from which we conclude $\boxed{\Pi^\dagger \hat{x}\, \Pi = -x = -\hat{x}}$

Now for $\hat{p}$

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$$\langle p\rangle = -i\hbar\int_{-\infty}^{\infty}\psi^*(x)\frac{\partial}{\partial x}\psi(x)\,dx$$

$$\langle p\rangle \longrightarrow -i\hbar\int_{-\infty}^{\infty}\psi^*(-x)\frac{\partial}{\partial x}\psi(-x)\,dx$$

Do the exact same thing as before, i.e change of variables $y=-x$, and use $\frac{\partial}{\partial x}=-\frac{\partial}{\partial y}$, and then change variables again to $x=y$

$$\langle p\rangle \rightarrow +i\hbar\int_{-\infty}^{\infty}\psi^*(x)\frac{\partial}{\partial x}\psi(x)\,dx = -\langle p\rangle$$

Again, we could have written

$$\langle p\rangle \rightarrow -i\hbar\int_{-\infty}^{\infty}\psi^*(x)\left(\Pi^+\frac{\partial}{\partial x}\Pi\right)\psi(x)\,dx$$

leading to $\boxed{\Pi^+\hat{p}\,\Pi = -\hat{p}}$

(b) Scalar operator O_s : $\Pi^+ O_s\Pi = O_s$

act on both sides with Π

$$\Pi\Pi^+ O_s\Pi = \Pi O_s \quad \text{but } \Pi\Pi^+=1$$

$$O_s\Pi = \Pi O_s \implies \boxed{[O_s\,\Pi]=0}$$

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Pseudoscalar operator O_p

$\Pi^{\dagger} O_p \Pi = -O_p$ – Acting on both sides with Π: $O_p \Pi = -\Pi O_p \Rightarrow \boxed{\{O_p\ \Pi\} = 0}$

(c) This is exactly the same as part (b)

(d) Scalars: x^2, p^2, electric charge –

Pseudoscalars: dot products of vectors and pseudovectors

eg $\vec{r}\cdot\vec{L}$, $\vec{p}\cdot\vec{L}$

2 (a) $[L_z\ V_{\pm}] = [L_z\ V_x] \pm i[L_z\ V_y]$

$[L_z\ V_{\pm}] = i\hbar V_y \pm i\,[-i\hbar V_x] = \pm\hbar(V_x \pm i V_y)$

$$\boxed{[L_z\ V_{\pm}] = \pm\hbar V_{\pm}}$$

To find $[L^2\ V_{\pm}]$ lets find $[L^2\ V_x]$ and $[L^2\ V_y]$

$[L^2 V_x] = [L_x^2 + L_y^2 + L_z^2\ \ V_x] = [L_y^2\ V_x] + [L_z^2\ V_x]$

$= L_y[L_y\ V_x] + [L_y\ V_x]L_y + L_z[L_z\ L_x] + [L_z\ L_x]L_z$

$= L_y(-i\hbar V_z) + (-i\hbar V_z)L_y + L_z(i\hbar V_y) + i\hbar V_y L_z$

$= i\hbar(V_y L_z + L_z V_y - L_y V_z - V_z L_y)$

Similarly (using cyclical properties)

$$[L^2\ V_y] = i\hbar(-V_x L_z - L_z V_x + L_x V_z + V_z L_x]$$

$$[L^2\ V_\pm] = [L^2\ V_x] \pm i[L^2\ V_y]$$

$$[L^2\ V_\pm] = i\hbar(V_y L_z + L_z V_y - L_y V_z - V_z L_y]$$
$$\pm\hbar(V_x L_z + L_z V_x - L_x V_z - V_z L_x]$$

$$[L^2\ V_\pm] = \hbar(\pm V_x + iV_y)L_z + \hbar L_z(\pm V_x + iV_y)$$
$$-\hbar(\pm L_x + iL_y)V_z - \hbar V_z(\pm L_x + iL_y)$$

$$[L^2\ V_\pm] = \pm\hbar V_\pm L_z \pm \hbar L_z V_\pm \mp \hbar L_\pm V_z \mp \hbar V_z L_\pm$$

But $V_\pm L_z + L_z V_\pm = [L_z\ V_\pm] + 2L_z V_\pm$

$$V_\pm L_z + L_z V_\pm = \pm\hbar V_\pm + 2L_z V_\pm$$

Therefore

$$[L^2\ V_\pm] = \hbar^2 V_\pm \pm 2\hbar L_z V_\pm \mp \hbar[L_\pm V_z + V_z L_\pm]$$

But $L_\pm V_z + V_z L_\pm = L_x V_z + V_z L_x \pm i(L_y V_z + V_z L_y)$

$$= [L_x\ V_z] + 2V_z L_x \pm i([L_y\ V_z] + 2V_z L_y)$$

$$= -i\hbar V_y + 2V_z L_\pm \mp \hbar V_x$$

$$= \mp\hbar V_\pm + 2V_z L_\pm$$

$$[L^2\ V_\pm] = \hbar^2 V_\pm \pm 2\hbar L_z V_\pm \mp \hbar(\mp\hbar V_\pm + 2V_z L_\pm)$$

$$[L^2\, V_\pm] = \hbar^2 V_\pm \pm 2\hbar L_z V_\pm \pm \hbar^2 V_\pm \mp 2 V_z L_\pm$$

$$\boxed{[L^2\, V_\pm] = 2\hbar^2 V_\pm \pm 2\hbar V_\pm L_z \mp 2\hbar V_z L_\pm}$$

(This was a huge pain in the neck!)

(b) $L^2 V_+ = [L^2\, V_+] + V_+ L^2 =$

$L^2 V_+ = 2\hbar^2 V_+ + 2\hbar V_+ L_z - 2\hbar V_z L_+ + V_+ L^2$

Then let's see what the angular momentum is for the state $V_+ \psi$ given that

$L^2 \psi = l(l+1)\hbar^2 \psi$ and $L_z \psi = \hbar \psi$

and $L_+ \psi = 0$

$$L^2 (V_+ \psi) = [2\hbar^2 + 2\hbar^2 l + \hbar^2 l(l+1)](V_+ \psi)$$

$$L^2 (V_+ \psi) = \hbar^2 [2(l+1) + l(l+1)](V_+ \psi)$$

$$L^2 (V_+ \psi) = \hbar^2 (l+2)(l+1)(V_+ \psi)$$

So

$\phi = V_+ \psi$ is either zero or an eigenfunction of L^2 with $l' = l+1$

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Now let's check out $L_z(V_+\phi)$

$$L_zV_+ = [L_z\, V_+] + V_+L_z = \hbar V_+ + V_+L_z$$

Therefore

$$L_z(V_+\psi) = (\hbar + l\hbar)(V_+\psi) = (l+1)\hbar\,(V_+\psi)$$

So

$\boxed{\phi = V_+\psi \text{ is either zero or an eigenfunction of } L_z \text{ with } m = l+1}$

3 (a) $I(u) = \frac{1}{2}$ $I_3(u) = \frac{1}{2}$

$$I_3(uuu) = \frac{3}{2}$$

$$\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2} = (1\oplus 0)\otimes\frac{1}{2} = (1\otimes\frac{1}{2})\oplus\frac{1}{2}$$

$$= \frac{3}{2}\oplus\frac{1}{2}\oplus\frac{1}{2}$$

Since $I_3(uuu) = \frac{3}{2}$ I cannot be $\frac{1}{2}$

$$\Rightarrow \boxed{|\Delta^{++}\rangle = |\tfrac{3}{2}\ \tfrac{3}{2}\rangle}$$

(b) $|\Delta^{+}\rangle = |\frac{3}{2}\ \frac{1}{2}\rangle$

$$|p\pi^0\rangle = |\tfrac{1}{2}\ \tfrac{1}{2}\rangle|1\ 0\rangle = \sqrt{\tfrac{2}{3}}\,|\tfrac{3}{2}\ \tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}}\,|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

$$|n\pi^+\rangle = |\tfrac{1}{2}\, -\tfrac{1}{2}\rangle |1\ 1\rangle = \sqrt{\tfrac{1}{3}}\, |\tfrac{3}{2}\ \tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\, |\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

$\Rightarrow$ Since only the $I = 3/2$ final states contribute

$$\frac{A(\Delta^+ \to p\pi^0)}{A(\Delta^+ \to n\pi^+)} = \frac{\sqrt{2/3}}{\sqrt{1/3}} = \sqrt{2}$$

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$$\boxed{\frac{\text{prob}(\Delta^+ \to p\pi^0)}{\text{prob}(\Delta^+ \to n\pi^+)} = 2}$$

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$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(a)

$$\sigma_1^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_2^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(b)

$$\sigma_1\sigma_2 + \sigma_2\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 0 \checkmark$$

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$$\sigma_1\sigma_3+\sigma_3\sigma_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$= \begin{pmatrix}0&-1\\1&0\end{pmatrix}+\begin{pmatrix}0&1\\-1&0\end{pmatrix} = 0 \checkmark$$

$$\sigma_2\sigma_3+\sigma_3\sigma_2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$$

$$= \begin{pmatrix}0&i\\i&0\end{pmatrix}+\begin{pmatrix}0&-i\\-i&0\end{pmatrix} = 0 \checkmark$$

(c) $R = e^{i\theta\hat{n}\vec{S}/\hbar} = e^{i\frac{\theta}{2}\hat{n}\vec{\sigma}}$

we need to expand in a Taylor series

Look at $(\hat{n}\vec{\sigma})^2$

$$(\hat{n}\vec{\sigma})^2 = (n_1\sigma_1+n_2\sigma_2+n_3\sigma_3)(n_1\sigma_1+n_2\sigma_2+n_3\sigma_3)$$

$$= n_1n_2\{\sigma_1\,\sigma_2\} + n_1n_3\{\sigma_1\,\sigma_3\} + n_2n_3\{\sigma_2\,\sigma_3\}$$

$$+ n_1^2\sigma_1^2 + n_2^2\sigma_2^2 + n_3^2\sigma_3^2$$

$= n_1^2+n_2^2+n_3^2$ since $\{\sigma_i\,\sigma_j\} = 0$ for $i \neq j$

and $\sigma_i^2 =$

But $\hat{n}$ is a unit vector, so $n_1^2+n_2^2+n_3^2$

$\Rightarrow (\hat{n}\vec{\sigma})^2 = 1$

and all even powers of $\hat{n}\vec{\sigma}$ are $= \underline{1}$

Therefore all odd powers of $\hat{n}\vec{\sigma}$ are $= \hat{n}\vec{\sigma}$

Thus, the Taylor expansion of R gives

$$R = \underbrace{\sum_{n\,\text{even}} \frac{i^n}{n!}\left(\frac{\theta}{2}\right)^n}_{=\cos\theta/2} + i\hat{n}\vec{\sigma} \underbrace{\sum_{n\,\text{odd}} \frac{i^{n-1}}{n!}\left(\frac{\theta}{2}\right)^n}_{=\sin\theta/2}$$

$$\Rightarrow \boxed{R = \cos\theta/2 + i\sin\frac{\theta}{2}\,\hat{n}\vec{\sigma}}$$

(d) write $\uparrow = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad \downarrow = \begin{pmatrix}0\\1\end{pmatrix}$

$\sin\frac{\theta}{2} = \underline{1}$ $\quad \cos\theta/2 = 0$ $\quad \Rightarrow R = i\hat{n}\vec{\sigma}$

$\hat{n} = \hat{x}$ $\quad R = i\sigma_x = \begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$

$R\uparrow = \begin{pmatrix}0 & i\\ i & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\i\end{pmatrix} = i\downarrow$

$R\downarrow = \begin{pmatrix}0 & i\\ i & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}i\\0\end{pmatrix} = i\uparrow$

$\hat{n} = \hat{y}$ $\quad R = i\sigma_y = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$

$R\uparrow = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix} = -\downarrow$

$R\downarrow = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = +\uparrow$