Physics 115B, Problem Set 9

Not to be graded

1 Parity

We can classify operators depending on how they transform under parity, $\mathcal{O} \to \hat{\Pi}^\dagger \mathcal{O} \hat{\Pi}$.

(a) In one dimension, show that position and momentum operators are odd under parity.

(b) In three dimensions, operators that are invariant under parity *and* commute with $\vec{L}$ are called *scalar* operators, $\hat{\Pi}^\dagger \mathcal{O}_s \hat{\Pi} = \mathcal{O}_s$. Operators that flip sign under parity and commute with $\vec{L}$ are called *pseudoscalar* operators, $\hat{\Pi}^\dagger \mathcal{O}_p \hat{\Pi} = -\mathcal{O}_p$. Show that scalar operators satisfy $[\hat{\Pi}, \mathcal{O}_s] = 0$, while pseudoscalar operators satisfy $\{\hat{\Pi}, \mathcal{O}_p\} = 0$ (here $\{A, B\}$ denotes the anticommutator, $\{A, B\} = AB + BA$).

(c) Operators that flip sign under parity and whose commutators with $\vec{L}$ are like those of $\vec{L}$ itself are called *vector* operators, and $\hat{\Pi}^\dagger \mathcal{O}_v \hat{\Pi} = -\mathcal{O}_v$. Operators that are invariant under parity and whose commutators with $\vec{L}$ are like those of $\vec{L}$ itself are called *axial vector* operators, $\hat{\Pi}^\dagger \mathcal{O}_a \hat{\Pi} = \mathcal{O}_a$. Show that vector operators satisfy $\{\hat{\Pi}, \mathcal{O}_v\} = 0$, while axial vector operators satisfy $[\hat{\Pi}, \mathcal{O}_a] = 0$.

(d) Give an example of a scalar operator, a pseudoscalar operator, a vector operator, and an axial vector operator (feel free to Google around to find some examples).

2 Vector Operators

For any vector operator $\vec{V}$, one can define raising and lowering operators

$$V_\pm = V_x \pm i V_y$$

(a) Using the criterion for being a vector operator (namely $[L_i, V_j] = i\hbar \epsilon_{ijk} V_k$), show that

$$[L_z, V_\pm] = \pm \hbar V_\pm$$

$$[L^2, V_\pm] = 2\hbar^2 V_\pm \pm 2\hbar V_\pm L_z \mp 2\hbar V_z L_\pm$$

(b) Show that, if ψ is an eigenstate of L^2 and L_z with eigenvalues $\hbar^2\ell(\ell+1)$ and $\hbar\ell$, respectively, then either $V_+\psi$ is zero or $V_+\psi$ is also an eigenstate of L^2 and L_z with eigenvalues $\hbar^2(\ell+2)(\ell+1)$ and $(\ell+1)\hbar$, respectively. This means that acting on a state with maximal $m=\ell$, the operator V_+ either raises both the ℓ and m values by 1, or destroys the state.

3 More Isospin

Refer to Homework 8, Problem 4 for an explanation of isospin (I). Consider the Δ^{++} particle, which is a bound state of three u quarks (uuu).

(a) What are the I and I_3 quantum numbers of the Δ^{++} particle?

(b) By acting on the Δ^{++} state with the isospin lowering operator I_- we obtain a uud state of the same I but with I_3 lowered by one unit. This is the Δ^+ state. The decay of the Δ^+ into a nucleon (p or n) and a pion (π^+, π^0, π^-) is mediated by the strong interaction, which conserves I and I_3, as well as (of course) charge. What is the ratio of probabilities for the decays $\Delta^+ \to p\pi^0$ and $\Delta^+ \to n\pi^+$. (As explained in Homework 8, Problem 4, the probability of a particular process is proportional to the square of the matrix element of some operator sandwiched between the initial and the final state).

4 Pauli Matrices, Spin Rotations

Let σ_i be the Pauli matrices. The spin operators in matrix form for $s=\frac{1}{2}$ states are $S_i = \frac{\hbar}{2}\sigma_i$.

(a) Show that $\sigma_i^2 = 1$.

(b) Show that the Pauli matrices anticommute, ie, $\{\sigma_i\sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = 0$ if $i \neq j$.

(c) The unitary operator that generates a rotation by θ about an axis defined by the direction of the unit vector $\hat{n}$ is $R = \exp\Big(i\theta\hat{n}\cdot\vec{S}/\hbar\Big)$.

Show that $R = \cos\frac{\theta}{2} \; + i\hat{n}\cdot\vec{S}\sin\frac{\theta}{2}$.

(d) What happens if you rotate the spin up and spin down states by 180^o around the x-axis? Around the y-axis?