PHYSICS 115B HWK8

(1) $$\hat{U} = e^{i\hat{O}} = \sum_{n=0}^{\infty} \frac{i^n}{n!} O^n$$

$$\hat{U}^+ = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} O^n \qquad (\text{if } O^+ = O, (O^n)^+ = O^n)$$

$$\hat{U}^+ = e^{-i\hat{O}}.$$

At this point we could just say

$$\boxed{\hat{U}\hat{U}^+ = e^{i\hat{O}} e^{-i\hat{O}} = e^0 = \underline{1}}$$

but it would be nice to show it explicitly

$$\hat{U}\hat{U}^+ = \sum_{n=0}^{\infty} \frac{i^n}{n!} \hat{O}^n \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \hat{O}^m$$

$$\hat{U}\hat{U}^+ = \sum_{n,m=0}^{\infty} \frac{(i^{n+m})}{n!\,m!} \hat{O}^n (-\hat{O}^m)$$

Write $p = n+m$

$$\hat{U}\hat{U}^+ = \sum_{p=0}^{\infty} \sum_{n=0}^{p} \frac{(i)^p}{n!(p-n)!} \hat{O}^n (-\hat{O})^{p-n}$$

page 2

Binomiel

$$(a+b)^p = \sum_{n=0}^{p} \frac{p!}{n!(p-n)!} a^n b^{p-n}$$

Therefore $\hat{U}\hat{U}^\dagger = \sum_{p=0}^{\infty} \frac{(i)^p}{p!} (\hat{O}-\hat{O})^p$

$$(\hat{O}-\hat{O})^p = (\text{Zero})^p$$

$$\boxed{\hat{U}\hat{U}^\dagger = \sum_{p=0}^{\infty} \frac{(i\cdot \text{Zero})^p}{p!} = e^{i0} = 1}$$

2

$$|p'\rangle = T(a)|p\rangle = e^{-ia\hat{p}/\hbar}|p\rangle$$

$$|p'\rangle = e^{-iap/\hbar}|p\rangle$$

(Note the distinction between $\hat{p}$ which is an operator and p which is a number)

$$\langle x|p'\rangle = \psi_{p'}(x) = e^{-\frac{iap}{\hbar}}\langle x|p\rangle = e^{-\frac{iap}{\hbar}}\psi_p(x)$$

But also $\psi_{p'}(x) = T(a)\psi_p(x) = \psi_p(x-a)$

So $\psi_p(x-a) = e^{-iap/\hbar}\psi_p(x)$

Now set $x = a$

$$\psi_p(0) = e^{-iap/\hbar}\,\psi_p(a)$$

this is some constant $= A$

$$\psi_p(a) = A e^{+iap/\hbar}$$

relabeling $a = x$

$$\boxed{\psi_p(x) = A e^{ipx/\hbar}}$$

3 $\langle \psi_j | x^2 | \psi_j \rangle = \frac{2}{a}\int_0^a dx\, x^2 \sin^2 \frac{j\pi x}{a}$

$$= \frac{2}{a}\frac{a^3}{12}\left(2 - \frac{3}{\pi^2 j^2}\right) = \frac{a^2}{6}\left[2 - \frac{3}{\pi^2 j^2}\right]$$

$$\langle \psi_j | x | \psi_j \rangle = \frac{2}{a}\int_0^a dx\, x \sin^2 \frac{j\pi x}{a}$$

$$= \frac{2}{a}\frac{a^2}{4} = \frac{a}{2}$$

$$\langle \psi_j | x | \psi_k \rangle = \frac{2}{a}\int_0^a x \sin\frac{j\pi x}{a}\sin\frac{k\pi x}{a}\,dx$$

$$= \frac{2}{a}\,2a^2 jk\frac{\left[(-1)^{j+k} - 1\right]}{\pi^2 (j^2-k^2)^2}$$

$$= 4a\,jk\frac{\left[(-1)^{j+k} - 1\right]}{\pi^2 (j^2-k^2)^2}$$

If $j+k$ even

$$\langle \psi_j | x | \psi_k \rangle = 0$$

If $j+k$ odd

$$\langle \psi_j | x | \psi_k \rangle = -\frac{8ajk}{\pi^2 (j^2-k^2)^2}$$

(a)

$$\langle \Delta x^2 \rangle_d = \frac{a^2}{6}\left[2 - \frac{3}{\pi^2 j^2}\right] + \frac{a^2}{6}\left[2 - \frac{3}{\pi^2 k^2}\right] - 2\frac{a}{2}\frac{a}{2}$$

$$\langle \Delta x^2 \rangle_d = a^2\left[\frac{1}{3} + \frac{1}{3} - \frac{1}{2}\right] - \frac{a^2}{2\pi^2}\left[\frac{1}{j^2} + \frac{1}{k^2}\right]$$

$$\boxed{\langle \Delta x^2 \rangle_d = a^2\left[\frac{1}{6} - \frac{1}{2\pi^2}\frac{k^2 j^2}{k^2+j^2}\right]}$$

(b) If $j+k$ even $\boxed{\langle \Delta x^2 \rangle_+ = \langle \Delta x^2 \rangle_d}$

(c) If $j+k$ odd

$$\langle \Delta x^2 \rangle_+ = \langle \Delta x^2 \rangle_d - 2\,\frac{64 a^2 j^2 k^2}{\pi^4 (j^2-k^2)^4}$$

Page 5

$$\boxed{\langle \Delta x^2 \rangle_+ = a^2 \left[\frac{1}{6} - \frac{1}{2\pi^2} \frac{k^2 j^2}{k^2+j^2} - \frac{128\, k^2 j^2}{\pi^4 (j^2-k^2)^4} \right]}$$

(d) If $j+k$ even $\boxed{\langle \Delta x^2 \rangle_- = \langle \Delta x^2 \rangle_d}$

(e) If $j+k$ odd, same solution as (c) but with the sign in front of the last term flipped

$$\boxed{\langle \Delta x^2 \rangle_- = a^2 \left[\frac{1}{6} - \frac{1}{2\pi^2} \frac{k^2 j^2}{k^2+j^2} + \frac{128\, k^2 j^2}{\pi^4 (j^2-k^2)^4} \right]}$$

4 (a) $I = 1$ states

$$\boxed{\begin{aligned} \pi^+ &= |1\ 1\rangle = u\bar{d} \\ \pi^0 &= |1\ 0\rangle = \frac{u\bar{u} - d\bar{d}}{\sqrt{2}} \\ \pi^- &= |1\ {-1}\rangle = d\bar{u} \end{aligned}} \quad \text{or} \quad \boxed{\begin{aligned} &-u\bar{d} \\ &\frac{d\bar{d} - u\bar{u}}{\sqrt{2}} \\ &-d\bar{u} \end{aligned}}$$

$I = 0$ state $\boxed{|0\ 0\rangle = \frac{u\bar{u} + d\bar{d}}{\sqrt{2}}}$

(b)

page 6

(1) $pp \to d\pi^+$

$pp = |1\ 1\rangle$

$d\pi^+ = |1\ 1\rangle$

The amplitude $A_{pp} = A_1$ $\quad \sigma_{pp} \sim |A_1|^2$

where A_I is the amplitude for a given I

(2) $pn \to d\pi^0$

$pn = |\tfrac{1}{2}\ \tfrac{1}{2}\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}|1\ 0\rangle + \frac{1}{\sqrt{2}}|0\ 0\rangle$

$d\pi^0 = |1\ 0\rangle$

$A_{pn} = \frac{1}{\sqrt{2}} A_1 \quad \sigma_{pn} \sim \frac{1}{2}|A_1|^2$

(3) $nn \to d\pi^-$

$nn = |1\ -1\rangle$

$d\pi^- = |1\ -1\rangle \quad A_{nn} = A_1 \quad \sigma_{nn} \sim |A_1|^2$

$$\boxed{\sigma_{pp} : \sigma_{pn} : \sigma_{nn} = 1 : \tfrac{1}{2} : 1}$$

5 $T^{\dagger}(a)\,\hat{p}\,T(a)\,\psi(x)$

$$= T^{\dagger}(a)\,\hat{p}\,\psi(x-a)$$

$$= T(-a)\,\hat{p}\,\psi(x-a)$$

$$= \hat{p}\,T(-a)\,\psi(x-a)$$

$$= \hat{p}\,\psi(x-a+a) = \hat{p}\,\psi(x)$$

where I used the following

- $T(a)\psi(x) = \psi(x-a)$
- $T^{\dagger}(a) = T^{-1}(a) = T(-a)$
- $[T(a)\,\hat{p}] = 0$