Physics 115B, Problem Set 8

Due Friday, May 27, 5pm

Every problem is worth 10 points.
Every sub-question is worth the same, unless otherwise specified.

1 Unitarity and Hermiticity

Show that, for a Hermitian operator $\mathcal{O}$, the operator $U = \exp[i\mathcal{O}]$ is unitary. *Hint: first prove that the adjoint is given by* $U^\dagger = \exp[-i\mathcal{O}]$*, then prove* $U^\dagger U = 1$.

2 Momentum Eigenstates

Back in Physics 115A (in 1D) you took the wavefunction for a state of momentum p to be

$$\psi_p(x) \equiv \langle x|p\rangle = Ae^{ipx/\hbar}$$

where A is a normalization factor. Now we can *derive* this using the properties of the translation operator $T(a)$. If the state $|p\rangle$ is one of well-defined momentum p, how would you characterize the state $|p'\rangle \equiv T(a)|p\rangle$? Show that the position-space wavefunctions of these states are related by

$$\langle x|p'\rangle = \psi_{p'}(x) = e^{-iap/\hbar}\psi_p(x) = e^{-iap/\hbar}\langle x|p\rangle$$

and also

$$\psi_{p'}(x) = \psi_p(x-a)$$

Hence conclude that $\langle x|p\rangle = Ae^{ipx/\hbar}$.

3 Exchange Interaction is a Square Well

In this question, you will calculate the effect of the exchange interaction in an infinite square well. Consider two noninteracting particles of mass m. Recall the infinite square well eigenstates are

$$\psi_j(x) = \sqrt{\frac{2}{a}} \sin \frac{j\pi x}{a}$$

where $j = 1, 2, 3, \ldots$ and $x \in [0, a]$. Let the particles be in states ψ_j and ψ_k with $j \neq k$. We are interested in calculating the typical separation between the particles. As in lecture, we can do this as $\left\langle (x_1 - x_2)^2 \right\rangle$. For distinguishable particles, we get

$$\begin{aligned}
\left\langle (x_1 - x_2)^2 \right\rangle_d &= \langle \psi_k | \langle \psi_j | \, x_1^2 + x_2^2 - 2x_1 x_2 \, | \psi_j \rangle \, | \psi_k \rangle \\
&= \langle \psi_j | \, x_1^2 \, | \psi_j \rangle \langle \psi_k | \psi_k \rangle + \langle \psi_j | \psi_j \rangle \langle \psi_k | \, x_2^2 \, | \psi_k \rangle - 2 \langle \psi_j | \, x_1 \, | \psi_j \rangle \langle \psi_k | \, x_2 \, | \psi_k \rangle \\
&= \langle \psi_j | \, x^2 \, | \psi_j \rangle + \langle \psi_k | \, x_2^2 \, | \psi_k \rangle - 2 \langle \psi_j | \, x \, | \psi_j \rangle \langle \psi_k | \, x \, | \psi_k \rangle .
\end{aligned}$$

Note that the $|\psi_j\rangle$ are affected only by the x_1 and the $|\psi_k\rangle$ are affected only by the x_2. For a spatially (anti)symmetric wavefunction, we also get a contribution from cross-terms for the $\langle x_1 x_2 \rangle$ piece, giving

$$\left\langle (x_1 - x_2)^2 \right\rangle_\pm = \left\langle (x_1 - x_2)^2 \right\rangle_d \mp 2 |\langle \psi_j | \, x \, | \psi_k \rangle|^2$$

Note the following useful integrals.

$$\int_0^a dx \, x \sin^2 \left(\frac{j\pi x}{a} \right) = \frac{a^2}{4}$$

$$\int_0^a dx \, x^2 \sin^2 \left(\frac{j\pi x}{a} \right) = \frac{a^3}{12} \left(2 - \frac{3}{\pi^2 j^2} \right)$$

$$\int_0^a dx \, x \sin \left(\frac{j\pi x}{a} \right) \sin \left(\frac{k\pi x}{a} \right) = \frac{2a^2 jk \left[(-1)^{j+k} - 1 \right]}{\pi^2 (j^2 - k^2)^2}$$

Calculate $\left\langle (x_1 - x_2)^2 \right\rangle$ if the particles are

1. distinguishable.
2. in a symmetric spatial wave function, with $j + k$ even.
3. in a symmetric spatial wave function, with $j + k$ odd.
4. in an antisymmetric spatial wave function, with $j + k$ even.
5. in an antisymmetric spatial wave function, with $j + k$ odd.

The exchange interaction is also sometimes called the 'exchange force'. Is it actually a force? Why or why not?

4 Isospin

The concept of isospin was introduced in the 1930s. As far as the strong interaction is concerned, protons and neutrons are the same. They are both spin $\frac{1}{2}$ fermions, they have (almost) the same mass, and the same strong interaction properties. So, ignoring E&M effects, they can be thought of as the same particle (a "nucleon") with an additional intrinsic quantum number called "isospin" (I) which has the same algebraic properties as spin or angular moment.

The nucleon has $I = \frac{1}{2}$, the proton and the neutron are distinguished by the 3rd component of I, i.e., $I_3 = +\frac{1}{2}$ for the proton and $I_3 = -\frac{1}{2}$ for the neutron. We can write the states as
$|p\rangle = |\frac{1}{2} \quad +\frac{1}{2}\rangle_N$ and
$|n\rangle = |\frac{1}{2} \quad -\frac{1}{2}\rangle_N$
where the notation is the same as the one we used for angular momentum, and the subscript N (which we can also drop, as long as we know what we are talking about) indicates that these are nucleon states.
A more modern picture is in terms of quark up and down states u and d where now
$|u\rangle = |\frac{1}{2} \quad +\frac{1}{2}\rangle_q$ and
$|d\rangle = |\frac{1}{2} \quad -\frac{1}{2}\rangle_q$
where the subscript q indicates that these are q states.
Antiquarks also make up a $I = \frac{1}{2}$ doublet as
$-|d\rangle = |\frac{1}{2} \quad +\frac{1}{2}\rangle_{\bar{q}}$ and
$|u\rangle = |\frac{1}{2} \quad -\frac{1}{2}\rangle_{\bar{q}}$
(the minus sign is a technicality that is not important now).
(a) Quark charges are $q(u) = \frac{2}{3}e$ and $q(d) = -\frac{1}{3}e$; antiquarks have the opposite charge. Write down the various possible $|q\bar{q}\rangle$ states of total $I = 1$ and $I = 0$. The three $I = 1$ states correspond to the three pion states π^+, π^0, π^-.
(b) The deuteron (d) is a proton-neutron bound state of $I = 0$. Consider the following processes

1. $pp \to d\pi^+$
2. $pn \to d\pi^0$
3. $nn \to d\pi^-$

You will learn in 115C that for $ab \to c$ the probability of the process is given by the square of an amplitude $A_{ab\to c} = \langle c|S|ab\rangle$, where S here is the so-called S-matrix operator. In strong interactions, isospin (both I and I_3) are conserved, $A_{ab\to c}$ does depend on I but not on I_3. Find relationships between the cross-sections (ie, the relative probabilities) of the three processes above.

5 Translation of Momentum Operator

Let $T(a)$ be the 1D translation operator by a distance a. Show that the translation of the momentum operator $\hat{p}$ given by $\hat{p}' \equiv T(a)^\dagger \hat{p} T(a) = \hat{p}$.