PHYSICS 115B HOMEWORK 7

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1 (a) $$\boxed{\psi = \psi_a(x_1)\psi_b(x_2)\psi_c(x_3)}$$

(b) $$\boxed{\begin{aligned}\psi = \frac{1}{\sqrt{6}}\Big[&\psi_a(x_1)\psi_b(x_2)\psi_c(x_3) + \psi_a(x_1)\psi_c(x_2)\psi_b(x_3) \\ &+ \psi_b(x_1)\psi_a(x_2)\psi_c(x_3) + \psi_b(x_1)\psi_c(x_2)\psi_a(x_3) \\ &+ \psi_c(x_1)\psi_b(x_2)\psi_a(x_3) + \psi_c(x_1)\psi_a(x_3)\psi_b(x_2)\Big]\end{aligned}}$$

(c)

$$\boxed{\begin{aligned}\psi = \frac{1}{\sqrt{6}}\Big[&\psi_a(x_1)\psi_b(x_2)\psi_c(x_3) - \psi_a(x_1)\psi_c(x_2)\psi_b(x_3) \\ &- \psi_b(x_1)\psi_a(x_2)\psi_c(x_3) + \psi_b(x_1)\psi_c(x_2)\psi_a(x_3) \\ &- \psi_c(x_1)\psi_b(x_2)\psi_a(x_3) + \psi_c(x_1)\psi_a(x_3)\psi_b(x_2)\Big]\end{aligned}}$$

For (b), the answer is clearly symmetric.
For (c), copy the solution from (b). Then, start with the first term, change sign if there is one label

switch, keep the sign if there are two label switches

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(2) Let $\psi_0(x)$ and $\psi_1(x)$ be the wavefunctions corresponding to the ground state $(n=0)$ and the first excited state $(n=1)$ respectively (Energy $E=(n+\frac{1}{2})\hbar\omega$

Denote the spin eigenstates of the two combined particles as $|j\ m\rangle$

Note: for $s=1$, $1 \otimes 1 = 2 \oplus 1 \oplus 0$

symmetric

antisymmetric

while for $s=\frac{1}{2}$, $\frac{1}{2}\otimes\frac{1}{2} = 1 \oplus 0$

symmetric

(a) $\psi = \psi_0(x_1)\psi_0(x_2)\,|2\ m\rangle$ 5 spin states

or $\psi = \psi_0(x_1)\psi_0(x_2)\,|0\ 0\rangle$ 1 spin state

Lowest state

For the lowest state (bosons)

$$\boxed{\text{Degeneracy} = 5+1=6 \qquad E = \frac{\hbar\omega}{2}+\frac{\hbar\omega}{2} = \hbar\omega}$$

For the next excited state

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$$\psi = \frac{1}{\sqrt{2}}\left[\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2)\right]|2\ m\rangle \quad (5 \text{ spin states})$$

$$\psi = \frac{1}{\sqrt{2}}\left[\psi_0(x_1)\psi_1(x_2) - \psi_1(x_1)\psi_0(x_2)\right]|1\ m\rangle \quad (3 \text{ spin states})$$

$$\psi = \frac{1}{\sqrt{2}}\left[\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2)\right]|0\ 0\rangle \quad (1 \text{ spin state})$$

For the first excited state (bosons)

$$\boxed{\text{Degeneracy} = 5+3+8=9 \quad E = \frac{\hbar\omega}{2} + \frac{3\hbar\omega}{2} = 2\hbar\omega}$$

(b) Spin ½ fermions now

Lowest state

$$\psi = \psi_0(x_1)\psi_0(x_2)\,|0\ 0\rangle \quad (1 \text{ spin state})$$

$$\boxed{\text{Degeneracy} = 1, \ E = \hbar\omega}$$

Next state

$$\psi = \frac{1}{\sqrt{2}}\left[\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2)\right]|0\ 0\rangle \quad \underline{1 \text{ state}}$$

$$\psi = \frac{1}{\sqrt{2}}\left[\psi_0(x_1)\psi_1(x_2) - \psi_1(x_1)\psi_0(x_2)\right]|1\ m\rangle \quad \underline{3 \text{ states}}$$

For the 1st excited state, fermions

$$\boxed{\text{Degeneracy} = 4 \quad E = 2\hbar\omega}$$

③ Remember that for H the energy levels are $E = -\frac{R}{n^2}$ where R = Rydberg = $-13.6\,eV$

If Z is the charge of the nucleus, we get $E = -\frac{Z^2 R}{n^2}$ _ The ground state energy then since $Z^2 = 9$ is $\boxed{E = 9\left(R + R + \frac{R}{4}\right) = -275\,eV}$

Next, let me write the individual electron states as $\phi(n, m)$ where $n = 0, 1$ and m is the up ($\frac{1}{2}$) or down ($-\frac{1}{2}$) spin label

This is really the same problem as (1b) above, except that instead of a, b, c we have (n, m) and one of the n's is 2, and page 5

the other two are 1's. Then up to a constant factor of $\frac{1}{\sqrt{6}}$, we get

$$\Psi = \phi(1m_1)\phi(1m_2)\phi(2m_3) - \phi(1m_1)\phi(2m_3)\phi(1m_2)$$
$$- \phi(1m_2)\phi(1,m_1)\phi(2m_3) + \phi(1m_2)\phi(2m_3)\phi(1m_1)$$
$$- \phi(2m_3)\phi(1m_2)\phi(1m_1) + \phi(2m_3)\phi(1m_1)\phi(1m_2)$$

Clearly $m_1 \neq m_2$, but m_3 can be anything

Otherwise $\Psi = 0$

Possibilities then are

$m_1 = \frac{1}{2}$ $m_2 = -\frac{1}{2}$ $m_3 = \frac{1}{2}$ or $-\frac{1}{2}$

$m_1 = -\frac{1}{2}$ $m_2 = +\frac{1}{2}$ $m_3 = \frac{1}{2}$ or $-\frac{1}{2}$

$$\Rightarrow \boxed{\text{Degeneracy} = 4}$$

4 (a) $E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3}$

$\rho = \frac{N}{V} = \frac{\text{atoms}}{\text{mol}} \; \frac{\text{mol}}{g} \; \frac{g}{cm^3}$ (g = grams)

6×10^{3} $\frac{1}{108}$ $10.5\, g/cm^3$

Plugging numbers in, $\rho = 5.9\; 10^{22} \frac{1}{cm^3} = 5.9\; 10^{28} \frac{1}{m^3}$

using $\hbar = 1.06$ J sec

$m = 9.1\; 10^{-31}$ kg $\Longrightarrow$ $\boxed{E_F = 5.5\,eV}$

(b) $\frac{1}{2}mv^2 = E_F$ $\boxed{\frac{v}{c} = \frac{1}{c}\sqrt{\frac{2E_F}{m}} = 5\; 10^{-3}}$

not relativistic

(c) $T_F = \frac{E_F}{k_B}$ $\boxed{T_F = 6.4\; 10^{4}\, K}$

(d) $P = \frac{(3\pi^2)^{2/3}\hbar^2 \rho^{5/3}}{5m}$ $\boxed{P = 2.1\; 10^{10} \frac{N}{m^2}}$

5) The π^0's have no spin
therefore in order to conserve angular momentum the orbital angular momentum of the two π^0's must be $L=1$. The relative motion of the two π^0's in their CM frame is treated as it was that of a fictitious particle of coordinates $\vec{r} = \vec{r}_1 - \vec{r}_2$ - thus, under $1 \leftrightarrow 2$ interchange $\vec{r} \to -\vec{r}$ - the spherical harmonics for $L = 1$ are

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta \qquad Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi} \qquad Y_1^{-1} = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\phi}$$

Under $\vec{r} \to -\vec{r}$, all of $\cos\theta$, $\sin\theta\cos\phi$, $\sin\theta\sin\phi$ change sign - Therefore $Y_1^m(\theta,\phi)$ changes sign, ie, the spatial wavefunction (which is also the total wavefunction since there is no spin) is asymmetric and that cannot be for two identical bosons