PHYSICS 115B
HOMEWORK 6

(1) Eqtn 4.210

$$(-i\hbar\vec{\nabla} - q\vec{A})\psi = -i\hbar e^{ig}\vec{\nabla}\psi' \quad \text{with} \quad \psi = e^{ig}\psi'$$

and $g(\vec{r}) = \frac{q}{\hbar}\int_{\vec{r}_0}^{\vec{r}} \vec{A}(\vec{r}')\, d\vec{r}'$

Eqtn 4.211 is $(-i\hbar\vec{\nabla} - q\vec{A})^2\psi = -\hbar^2 e^{ig}\nabla^2\psi'$

Work on the LHS of 4.211 with $\psi = e^{ig}\psi'$

$$(-i\hbar\vec{\nabla} - q\vec{A})(-i\hbar\vec{\nabla} - q\vec{A})e^{ig}\psi' =$$

$$(-i\hbar\vec{\nabla} - q\vec{A})(-i\hbar e^{ig}\vec{\nabla}\psi' - i\hbar(ie^{ig})\psi'\vec{\nabla}g - e^{ig}q\vec{A}\psi')$$

But $\vec{\nabla}g = \frac{q}{\hbar}\vec{A}$, therefore the last two terms in the equation above cancel -

LHS of 4.211 $= (-i\hbar\vec{\nabla} - q\vec{A})(-i\hbar e^{ig}\vec{\nabla}\psi')$

$$= -\hbar^2\vec{\nabla}(e^{ig}\vec{\nabla}\psi') + i\hbar q\vec{A}\vec{\nabla}\psi'$$

$$= -\hbar^2 e^{ig}\nabla^2\psi' - \hbar^2\vec{\nabla}\psi' \cdot ie^{ig}\vec{\nabla}g + i\hbar q\vec{A}\vec{\nabla}\psi'$$

$$= -\hbar^2 e^{ig}\nabla^2\psi' - i\hbar^2 e^{ig}\vec{\nabla}\psi'\frac{q}{\hbar}\vec{A} + i\hbar q\vec{A}\vec{\nabla}\psi'$$

$$\boxed{\text{LHS of } 4.211 = -\hbar^2 e^{ig}\nabla^2\psi' = \text{RHS of } 4.211}$$

2 (a) the state is $|1\ -1\rangle = a|\tfrac{3}{2}\ -\tfrac{3}{2}\rangle|\tfrac{1}{2}\ +\tfrac{1}{2}\rangle$

$b|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$

Looking up the CG in the table I find

$a = -\sqrt{3}/2 \quad b = 1/2$

$\Rightarrow$ Measuring S_z on the spin $\tfrac{3}{2}$ particle I would get

$$\boxed{S_z = -\frac{3\hbar}{2}\ \text{prob} = 3/4 \ \text{ or } \ S_z = \frac{-\hbar}{2}\ \text{prob} = \frac{1}{4}}$$

(b) The state now is

$$|\tfrac{3}{2}\ -\tfrac{3}{2}\rangle|\tfrac{1}{2}\ +\tfrac{1}{2}\rangle = a|2\ -1\rangle + b|1\ -1\rangle$$

Looking up the CG: $a = 1/2 \quad b = -\sqrt{3}/2$

$$\boxed{S^2 = 2(2+1)\hbar^2 = 6\hbar^2\ \text{prob} = 1/4 \ \text{ or } \ S^2 = 1(1+1)\hbar^2 = 2\hbar^2\ \text{prob} = 3/4}$$

(c) $|2\ -1\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle = a|\tfrac{5}{2}\ -\tfrac{1}{2}\rangle + b|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle$

with $a = \sqrt{2/5}$ $\quad b = -\sqrt{3/5}$

$$\boxed{\begin{array}{l} J^2 = \frac{5}{2}\left(\frac{5}{2}+1\right)\hbar^2 = \frac{35}{4}\hbar^2 \quad \text{prob} = 2/5 \\ J^2 = \frac{3}{2}\left(\frac{3}{2}+1\right)\hbar^2 = \frac{15\hbar^2}{4} \quad \text{prob} = 3/5 \end{array}}$$

$\vec{J} = \vec{L} + \vec{S}$

(d) $|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle = a|2\ 0\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle + b|2\ -1\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$

with $a = \sqrt{2/5}$ $\quad b = -\sqrt{3/5}$

$$\boxed{\text{prob of } S_z = -\frac{\hbar}{2} \text{ is } 2/5}$$

3 Let's take $\hat{a} = \hat{z}$ $\quad \hat{b} = (\sin\theta\ 0\ \cos\theta)$

Using eqtn 4.155, the up/down eigenspinors in the direction of $\hat{b}$, denoted by $\Uparrow, \Downarrow$ can be expressed in terms of the eigenspinors in the direction of z ($\uparrow$ and $\downarrow$) as

$$\Uparrow = \cos\frac{\theta}{2}\uparrow + \sin\frac{\theta}{2}\downarrow \qquad \Downarrow = \sin\frac{\theta}{2}\uparrow - \cos\frac{\theta}{2}\downarrow$$

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Turning this around

$$\uparrow = \cos\frac{\theta}{2}\Uparrow - \sin\frac{\theta}{2}\Downarrow \qquad \downarrow = \sin\frac{\theta}{2}\Uparrow + \cos\frac{\theta}{2}\Downarrow$$

The state $|0\,0\rangle = \frac{1}{\sqrt{2}}\left[\uparrow(1)\downarrow(2) - \downarrow(1)\uparrow(2)\right]$

Writing $\uparrow(2)$ and $\downarrow(2)$ in terms of $\Uparrow(2)$ and $\Downarrow(2)$ and dropping the (1) or (2) labels we have

$$|0\,0\rangle = \frac{1}{\sqrt{2}}\left[\sin\frac{\theta}{2}\uparrow\Uparrow + \cos\frac{\theta}{2}\uparrow\Downarrow - \cos\frac{\theta}{2}\downarrow\Uparrow + \sin\frac{\theta}{2}\downarrow\Downarrow\right]$$

Then the expectation value is

$$\langle S_{1a}S_{2b}\rangle = \frac{\hbar^2}{2}\left[\frac{1}{4}\sin^2\frac{\theta}{2} - \frac{1}{4}\cos^2\frac{\theta}{2} - \frac{1}{4}\cos^2\frac{\theta}{2} + \frac{1}{4}\sin^2\frac{\theta}{2}\right]$$

$$\boxed{\langle S_{1a}S_{2b}\rangle = \frac{\hbar^2}{8}\left[\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}\right] = -\frac{\hbar^2}{4}\cos\theta}$$

BONUS DERIVATION OF EQUATION 4.155 FROM PROBLEM 4.33 AT THE END page5

(4) $C \to AB$

(a) $S(C) = 0$, $S(A) = S(B) = 1/2$

$$S(A) \otimes S(B) \otimes L(AB) =$$

$$\tfrac{1}{2} \otimes \tfrac{1}{2} \otimes L(AB) =$$

$$(0 \oplus 1) \otimes L(AB) =$$

$$L(AB) \oplus L(AB)+1 \oplus L(AB) \oplus L(AB)-1$$

Since the total must be 0, $\boxed{L(AB) = 0 \text{ or } 1}$

(b) $\tfrac{1}{2} \otimes 1 \otimes L = \left(\tfrac{3}{2} \oplus \tfrac{1}{2}\right) \otimes L$

This must give $3/2$.

$L=0$ works since $3/2 \otimes 0 = 3/2$

$L=1$ works since $3/2 \otimes 1 = 5/2 \oplus 3/2 \oplus 1/2$

and $1/2 \otimes 1 = 3/2 \oplus 1/2$

$L=2$ works since $3/2 \otimes 2 = 7/2 \oplus 5/2 \oplus 3/2 \oplus 1/2$

$1/2 \otimes 2 = 5/2 \oplus 3/2 \oplus 1/2$

$L=3$ works since

$$\tfrac{3}{2} \otimes 3 = \tfrac{9}{2} \oplus \tfrac{7}{2} \oplus \tfrac{5}{2} \oplus \tfrac{3}{2}$$

$$\tfrac{1}{2} \otimes 3 = \tfrac{7}{2} \oplus \tfrac{5}{2}$$

$L=4$ does not work $\quad \tfrac{3}{2} \otimes 4 = \tfrac{11}{2} \oplus \tfrac{9}{2} \oplus \tfrac{7}{2} \oplus \tfrac{5}{2}$

$$\tfrac{1}{2} \otimes 4 = \tfrac{9}{2} \oplus \tfrac{7}{2}$$

$$\boxed{\text{Allowed values } L = 0, 1, 2, 3}$$

(c) $n \to p e^-$

The final state has $S = 0, 1$. Since L is integer $S \otimes L$ = integer – But n has $S = 1/2$

5 (a) $\vec{A} = -\frac{1}{2} \vec{r} \times \vec{B}_0$

Remember that $\vec{\nabla} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{\nabla} \times \vec{a}) - \vec{a} \cdot (\vec{\nabla} \times \vec{b})$

Therefore $\vec{\nabla} \cdot \vec{A} = -\frac{1}{2} \vec{B} \cdot (\vec{\nabla} \times \vec{r}) + \vec{r} \cdot (\vec{\nabla} \times \vec{B}_0)$

$$\vec{\nabla} \cdot \vec{A} = -\tfrac{1}{2} \vec{B} \cdot (\vec{\nabla} \times \vec{r}) \quad \text{since } \vec{B}_0 \text{ constant}$$

But $\vec{\nabla} \times \vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = 0$

$$\Rightarrow \boxed{\vec{\nabla} \vec{A} = 0}$$

(b) $\vec{B} = \text{curl}\,\vec{A} = -\frac{1}{2}\,\text{curl}\,(\vec{r} \times \vec{B}_0)$

Use the fact that

$$\text{curl}\,(\vec{a} \times \vec{b}) = \vec{a}\,\text{div}\,\vec{b} - \vec{b}\,\text{div}\,\vec{a} + (\vec{b} \cdot \vec{\nabla})\vec{a} - (\vec{a} \cdot \vec{\nabla})\vec{b}$$

$$\vec{B} = -\frac{1}{2}\Big[\underbrace{\cancel{\vec{r}\,\vec{\nabla}\vec{B}_0}}_{=0 \text{ since } \vec{B}_0 = \text{const}} - \vec{B}_0\,\vec{\nabla}\vec{r} + (\vec{B}_0\,\vec{\nabla})\vec{r} - \underbrace{\cancel{(\vec{r}\,\vec{\nabla})\vec{B}_0}}_{=0 \text{ since } \vec{B}_0 = \text{const}}\Big]$$

$$\vec{\nabla}\vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$(\vec{B}_0 \vec{\nabla})\vec{r} = \left[B_{0x}\frac{\partial}{\partial x} + B_{0y}\frac{\partial}{\partial y} + B_{0z}\frac{\partial}{\partial z}\right]\left[x\hat{x} + y\hat{y} + z\hat{z}\right]$$

$$(\vec{B}_0 \vec{\nabla})\vec{r} = B_{0x}\hat{x} + B_{0y}\hat{y} + B_{0z}\hat{z} = \vec{B}_0$$

$$\rightarrow \boxed{\vec{B} = -\frac{1}{2}\left[-3\vec{B}_0 + \vec{B}_0\right] = \vec{B}_0}$$

(C) $H = \frac{1}{2m}\left[\vec{p} - q\vec{A}\right]^2 + q\varphi + V(r) - \gamma \vec{B}_0 \vec{S}$

$$H = \frac{p^2}{2m} + \frac{q^2}{2m}A^2 - \frac{q}{2m}\left[\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}\right] + q\varphi + V(r) - \gamma\vec{B}_0\cdot\vec{S}$$

$$\vec{p}\cdot\vec{A} = -i\hbar \underbrace{\cancel{\vec{\nabla}\vec{A}}}_{=0} - i\hbar\vec{A}\vec{\nabla} = -i\hbar\vec{A}\vec{\nabla} = \vec{A}\cdot\vec{p}$$

Therefore

$$H = \frac{p^2}{2m} + \frac{q^2}{2m}A^2 - \frac{q}{2m}\vec{A}\cdot\vec{p} + q\varphi + V(r) - \gamma\vec{B}_0\cdot\vec{S}$$

We need $\vec{A}\cdot\vec{p}$ and A^2

$$\vec{A}\cdot\vec{P} = -\frac{1}{2}(\vec{r}\times\vec{B}_0)\cdot\vec{P} = +\frac{1}{2}(\vec{B}_0\times\vec{r})\cdot\vec{p}$$

$$\vec{A}\cdot\vec{p} = \frac{1}{2}\vec{B}_0\cdot(\vec{r}\times\vec{p}) = \frac{1}{2}\vec{B}_0\cdot\vec{L}$$

and, using $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$

$$A^2 = \frac{1}{4}(\vec{r}\times\vec{B}_0)(\vec{r}\times\vec{B}_0) = r^2B_0^2 - (\vec{r}\cdot\vec{B}_0)^2$$

Inserting the expressions for A^2 and $\vec{A}\cdot\vec{p}$ into H, we get

$$\boxed{H = \frac{p^2}{2m} + V(r) + q\varphi - \frac{q}{2m}\vec{B}_0\cdot\vec{L} - \gamma\vec{B}_0\cdot\vec{S} + \frac{q^2}{8m}\left[r^2B_0^2 - (\vec{r}\cdot\vec{B}_0)^2\right]}$$

BONUS SOLUTION TO PROBLEM 4.33 IN GRIFFITHS:

Construct the spin operator in direction of $\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

$$S_r = \vec{S}\cdot\hat{r} = S_x\sin\theta\cos\phi + S_y\sin\theta\sin\phi + S_z\cos\theta$$

Shorthand now: $s = \sin\theta$ $c = \cos\theta$

$s' = \sin\phi$ $c' = \cos\phi$

$$S_r = \frac{\hbar}{2}\begin{pmatrix} 0 & sc' \\ sc' & 0 \end{pmatrix} + \frac{\hbar}{2}\begin{pmatrix} 0 & -iss' \\ iss' & 0 \end{pmatrix} + \frac{\hbar}{2}\begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}$$

$$\boxed{S_r = \frac{\hbar}{2}\begin{pmatrix} c & s e^{-i\phi} \\ e^{i\phi} s & -c \end{pmatrix}}$$

where $c = \cos\theta$
$s = \sin\theta$

Now we need the eigenvectors of this matrix

eigenvalues $= \lambda$

$$\det \begin{vmatrix} \frac{\hbar}{2}c - \lambda & \frac{\hbar}{2} s e^{-i\phi} \\ \frac{\hbar}{2} s e^{i\phi} & -\frac{\hbar}{2} c - \lambda \end{vmatrix} = 0$$

$$\lambda^2 - \frac{\hbar^2}{4} c^2 - \frac{\hbar^2}{4} s^2 = 0 \qquad \lambda = \pm \frac{\hbar}{2}$$

(what a surprise!)

Now find the eigenvectors

$$\frac{\hbar}{2}\begin{pmatrix} c & s e^{-i\phi} \\ s e^{i\phi} & -c \end{pmatrix} = \pm \frac{\hbar}{2}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{cases} c\alpha + s e^{-i\phi}\beta = \pm\alpha \\ s e^{i\phi}\alpha - c\beta = \pm\beta \end{cases}$$

The top equation gives $\beta = e^{i\phi}\left(\frac{1-c}{s}\right)\alpha$

or $\beta = -e^{i\phi}\frac{(1+c)}{s}\alpha$

But $1-c = 1-\cos\theta = 2\sin^2\theta/2$

$1+c = 1+\cos\theta = 2\cos^2\theta/2$

$s = \sin\theta = 2\sin\theta/2 \cos\theta/2$

So the two solutions for β are

$$\beta = e^{i\phi}\frac{\sin\theta/2}{\cos\theta/2}\alpha \qquad \beta = -e^{i\phi}\frac{\cos\theta/2}{\sin\theta/2}$$

So up to a normalization constant the eigenvectors are

$$A\begin{pmatrix} 1 \\ e^{i\phi}\tan\theta/2 \end{pmatrix} \qquad B\begin{pmatrix} 1 \\ -e^{i\phi}\cot\theta/2 \end{pmatrix}$$

Normalize them now

$$|A|^2\left(1+\tan^2\theta/2\right) = 1 \qquad |B|^2\left(1+\cot^2\theta/2\right) = 1$$

$$|A|^2\frac{1}{\cos^2\theta/2} = 1 \qquad |B|^2\frac{1}{\sin^2\theta/2} = 1$$

$|A|^2 = \cos^2\theta/2$ $\qquad$ $|B|^2 = \sin^2\theta/2$

$A = e^{i\alpha}\cos\theta/2$ $\qquad$ $B = e^{i\beta}\sin\theta/2$ $\qquad$ (α, β arbitrary)

eigenvectors then are

$$\boxed{\begin{array}{l} e^{i\alpha}\begin{pmatrix}\cos\theta/2 \\ e^{i\phi}\sin\theta/2\end{pmatrix} \qquad e^{i\beta}\begin{pmatrix}\sin\theta/2 \\ -e^{i\phi}\cos\theta/2\end{pmatrix} \\ \text{to get the answer in the book} \\ \text{I can choose } \alpha = 0 \quad \beta = -\phi \\ \begin{pmatrix}\cos\theta/2 \\ e^{i\phi}\sin\theta/2\end{pmatrix} \qquad \begin{pmatrix}e^{-i\phi}\sin\theta/2 \\ -\cos\theta/2\end{pmatrix} \end{array}}$$