Physics 115B, Problem Set 6

Due Friday, May 13, 5pm

Every problem is worth 10 points.
Every sub-question is worth the same, unless otherwise specified.

1 Vector potential

Derive Equation 4.211 in Griffiths, starting with Equation 4.210.

2 Clebsch-Gordanalysis

(a) A particle of spin $s_1 = 3/2$ and a particle of spin $s_2 = 1/2$ are at rest in a configuration where the total spin of the two particles is 1 and a measurement of the total spin of the two particles in the $\hat{z}$ direction yields $-\hbar$. If you then measured the spin of the spin-3/2 particle in the $\hat{z}$ direction, what values might you obtain, and with what probabilities?

(b) Imagine that the outcome of your measurement in part (a) was $-3\hbar/2$. You subsequently measure the total spin of the two-particle system. What values might you obtain, and with what probabilities?

(c) An electron with spin up in the $\hat{z}$ direction is in the state $\psi_{4,2,-1}$ of the hydrogen atom. If you measured the total angular momentum (orbital angular momentum plus spin) of the electron, what values might you obtain, and with what probabilities?

(d) Imagine that the outcome of your measurement of the total angular momentum in part (c) found $j = 3/2$. You subsequently measure the spin of the electron in the $\hat{z}$ direction. What is the probability that your result is $-\hbar/2$?

3 Spin Angles

Suppose two spin-1/2 particles are known to be in the $|00\rangle$ state in the coupled basis (this is known as the "spin singlet" state). Let $S_{1,a}$ be the component of the spin of particle 1 in the direction of the vector $\hat{a}$. Similarly, let $S_{2,b}$ be the component of the spin of particle 2 in the direction of the vector $\hat{b}$. Show that

$$\langle S_{1,a} S_{2,b} \rangle = -\frac{\hbar^2}{4} \cos\theta$$

where θ is the angle between $\hat{a}$ and $\hat{b}$.
Hint: since we are working in the $|00\rangle$ state, there is clearly no preferred direction in space. Thus, it may help to take one of $\hat{a}$ or $\hat{b}$ in a conveniently simple direction. Also, you may want to use equation 4.155 from Problem 4.33. (I will include in my solutions also a solution to problem 4.33 – you are welcome to try it yourself, but you do not have to turn it in).

4 Decaying Particle

This problem involves the decay of an unstable particle C to particles A and B, in which total angular momentum is conserved. In the rest frame of C, the total angular momentum $\vec{J} = \vec{S}_C$ is just the spin of the particle C. After the decay, the total angular momentum consists of three terms,

$$\vec{J} = \vec{S}_A + \vec{S}_B + \vec{L}$$

where $\vec{S}_A$ is the spin of particle A, $\vec{S}_B$ is the spin of particle B, and $\vec{L}$ is the orbital angular momentum between A and B. Conservation of angular momentum in this decay means that if the initial state is an eigenstate of J^2 and J_z, then the final state is also an eigenstate with the same eigenvalues.

(a) Consider the case where C is a spin-0 particle and A, B are both spin-1/2 particles ($s_A = s_B = 1/2$). What values of the orbital angular momentum ℓ are consistent with angular momentum conservation? (3 points)

(b) Repeat the above problem, but now where C is a spin-3/2 particle, A is a spin-1/2 particle, and B is a spin-1 particle. (3 points)

(c) There are certain processes for which a two-body decay is forbidden. Explain why a neutron n cannot decay to a proton p and an electron e^- (all spin-1/2 fermions), despite this being consistent with energy and charge conservation. (4 points)

5 Particle in a field

Consider a particle with charge q, mass m, and spin s, in a **uniform** magnetic field $\vec{B}_0$ and subject to a central potential $V(r)$. The vector potential can be chosen as

$$\vec{A} = -\frac{1}{2}\vec{r} \times \vec{B}_0$$

(a) Verify that this potential satisfies $\nabla \cdot \vec{A} = 0$, i.e, we are working in the Coulomb gauge. (2 points)

(b) Verify that the vector potential produces a uniform magnetic field $\vec{B}_0$. (3 points)

(c) Show that the Hamiltonian can be written as

$$H = \frac{p^2}{2m} + V(r) + q\varphi - \vec{B}_0 \cdot (\gamma_0 \vec{L} + \gamma \vec{S}) + \frac{q^2}{8m}\left[r^2 B_0^2 - (\vec{r} \cdot \vec{B}_0)^2\right]$$

where $\gamma_0 = q/2m$ is the gyromagnetic ratio for orbital motion. (5 points)

Note: the term linear in $\vec{B}_0$ makes it energetically favorable to for the magnetic moments (orbital and spin) to align with the magentic field (**paramagnetism**). The terms quadratic in B_0 lead to the opposite effect (**diamagnetism**). This last statement is not immediately obvious.