Physics 115B HWK3

1 (a) $L_+|l\,m\rangle = \alpha_+(l,m)\,|l\ m+1\rangle$

Inner product of both sides:

$$\langle l m|L_- L_+|l m\rangle = |\alpha_+|^2 \langle l\ m{+}1|l\ m{+}1\rangle$$

Since $L_- L_+ = L^2 - L_z^2 + \hbar L_z$

$$l(l+1)\hbar^2 - \hbar^2 m^2 + \hbar^2 m = |\alpha_+|^2$$

where I also used the normalization $\langle lm|lm\rangle = 1$ and $\langle l m{+}1|l\, m{+}1\rangle = 1$

Thus, up to a phase $\boxed{\alpha_+(l,m) = \hbar\sqrt{l(l+1) - m(m+1)}}$

Similarly taking inner products of both sides of

$$L_-|l,m\rangle = \alpha_-(l,m)\,|l\ m-1\rangle$$

and using $L_+ L_- = L^2 - L_z^2 - \hbar L_z$ we find

$$l(l+1)\hbar^2 - \hbar^2 m^2 - \hbar^2 m = |\alpha_-|^2$$

which leads to $\boxed{\alpha_-(l,m) = \hbar\sqrt{l(l-1) - m(m-1)}}$

also up to a phase

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② (a) Do it using ε_{ijk} symbol

using $L_3 = \varepsilon_{ij3} x_i p_j$

$$[L_3\, x_k] = \varepsilon_{ij3} x_i [p_j\, x_k] = -i\hbar\, \delta_{jk}\, \varepsilon_{ij3}\, x_i$$

$$[L_3\, x_k] = -i\hbar\, \varepsilon_{ik3}\, x_i$$

$k=1 \Rightarrow [L_3\, x_1] = -i\hbar\, \varepsilon_{i13}\, x_i = +i\hbar x_2 = +i\hbar y$

$k=2 \Rightarrow [L_3\, x_2] = -i\hbar\, \varepsilon_{i23}\, x_i = -i\hbar x_1 = -i\hbar x$

$k=3 \Rightarrow [L_3\, x_3] = -i\hbar\, \varepsilon_{i33}\, x_i = 0$

Now the commutators with momentum

$$[L_3\, p_k] = \varepsilon_{ij3} [x_i\, p_j\, p_k] = \varepsilon_{ij3}\, p_j\, [x_i\, p_k]$$

$$[L_3\, p_k] = \varepsilon_{ij3}\, p_j\, i\hbar\, \delta_{ik} = i\hbar\, \varepsilon_{kj3}\, p_j$$

$k=1 \Rightarrow [L_3\, p_1] = i\hbar\, \varepsilon_{1j3}\, p_j = i\hbar p_2 = i\hbar p_y$

$k=2 \Rightarrow [L_3\, p_2] = i\hbar\, \varepsilon_{2j3}\, p_j = -i\hbar p_1 = -i\hbar p_x$

$k=3 \Rightarrow [L_3\, p_3] = i\hbar\, \varepsilon_{3j3}\, p_j = 0$

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(b) $[L_z\ L_x] = [L_z\ yp_z - zp_y] =$

$= [L_z\ yp_z] - [L_z\ zp_y] = L_z y p_z - y p_z L_z - L_z z p_y + z p_y L_z$

Both z and p_z commute with L_z. Thus

$$[L_z\ L_x] = (L_z y - y L_z) p_z - z(L_z p_y - p_y L_z)$$

$$= [L_z\ y] p_z - z[L_z\ p_y] =$$

$$= -i\hbar x p_z - z(-i\hbar p_x)$$

$$\boxed{[L_z\ L_x] = i\hbar(z p_x - x p_z) = i\hbar L_y}$$

(c) $[L_z\ r^2] = [L_z\ x^2] + [L_z\ y^2] + [L_z\ z^2] =$

$= [L_z x]x + x[L_z\ x] + y[L_z\ y] + [L_z\ y]y + z[L_z z] + [L_z z]z$

$= i\hbar(yx + xy - yx - xy + 0 + 0)$

$$\boxed{[L_z\ r^2] = 0}$$

Similarly $[L_z\ p^2] = p_x[L_z\ p_x] + [L_z\ p_x]p_x + p_y[L_z p_y]$

$+ [L_z\ p_y]p_y + [L_z\ p_z]p_z + p_z[L_z\ p_z]$

$$[L_z\ p^2] = i\hbar\left(P_xP_y + P_yP_x - P_yP_x - P_xP_y + 0 + 0\right)$$

$$\boxed{[L_z\ p^2] = 0}$$

③ Let's look at $\vec{L}$ component by component

(a)

$$\frac{d\langle L_i\rangle}{dt} = \frac{i}{\hbar}\langle [H\ L_i]\rangle$$

$$[H\ L_i] = \frac{1}{2m}\underbrace{[p^2\ L_i]} + [V\ L_i] = [V\ L_i]$$

This is zero from the previous problem

$$[V\ L_i] = [V\ \varepsilon_{ijk}x_jP_k] = \varepsilon_{ijk}x_j[V\ P_k]$$

$$[V\ L_i] = +i\hbar\underbrace{\varepsilon_{ijk}x_j\frac{\partial V}{\partial x_k}}_{=\text{ ith component of }\vec{r}\times\vec{\nabla}V} = i\hbar[\vec{r}\times\vec{\nabla}V]_i$$

Thus $[V \vec{L}] = i\hbar [\vec{r} \times \vec{\nabla} V]$

using the defintion of torque

$$[V \vec{L}] = -i\hbar \vec{N}$$

Since $[H \vec{L}] = [V \vec{L}]$, $[H \vec{L}] = -i\hbar \vec{N}$

$$\boxed{\frac{d\langle \vec{L} \rangle}{dt} = \frac{i}{\hbar}\langle [H\ L] \rangle = \langle \vec{N} \rangle}$$

(b) In spherical coordinates

$$\vec{\nabla} V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}$$

if $V(\vec{r}) = V(r)$, ie spherically symmetric

then $\vec{\nabla} V = \frac{dV}{dr}\hat{r}$ Then $\boxed{\vec{N} = -\vec{r} \times \vec{\nabla} V = 0}$