Physics 115B, Problem Set 3

Due Friday, April 22, 5pm

Every problem is worth 10 points.
Every sub-question is worth the same, unless otherwise specified.

1 Angular Ladder

As we have seen in lecture, the raising and lowering operators for angular momentum change the value of m by one unit. However, if the eigenstates $|\ell, m\rangle$ are normalized, we are only guaranteed that $L_\pm |\ell, m\rangle \propto |\ell, m \pm 1\rangle$, up to scaling factors $\alpha_\pm(\ell, m)$:

$$L_+ |\ell, m\rangle = \alpha_+(\ell, m) |\ell, m+1\rangle \qquad L_- |\ell, m\rangle = \alpha_-(\ell, m) |\ell, m-1\rangle$$

Determine $\alpha_\pm(\ell, m)$ assuming the states $|\ell, m\rangle$ are normalized.
Hint: Recall that $L_+^\dagger = L_-$ and take the inner product of both sides of the first equation.

2 Angular Algebra

(a) Starting with the commutation relations for position and momentum in Euclidean coordinates, work out the commutators $[L_z, x], [L_z, p_x], [L_z, y], [L_z, p_y], [L_z, z], [L_z, p_z]$. (4 points)

(b) Use your results from part (a) and the definitions $L_x = yp_z - zp_y, L_y = zp_x - xp_z$ to verify that $[L_z, L_x] = i\hbar L_y$. (3 points)

(c) Find the commutators $[L_z, r^2]$ and $[L_z, p^2]$, where $r^2 = x^2 + y^2 + z^2$ and $p^2 = p_x^2 + p_y^2 + p_z^2$. (3 points)

Now here is the important point. In part (c) you have shown that L_z commutes with r^2 and p^2. Since there is nothing special about the z-axis, the same would hold for L_x and L_y and thus for L^2. The hamiltonian H for a particle in a spherically symmetric potential is a function of $r = \sqrt{r^2}$ and p^2, and so it commutes with L^2 as well as all of L_x, L_y, and L_z individually. However the individual components of angular momentum do not commute with each other, ie, $[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$. Therefore H, L^2, and one of the L_i's form a set of compatible observables. The conventional choice is to label the eigenstates in terms of the eigenvalues of H, L^2, and L_z.

3 Angular Ehrenfest

Remember the Ehrenfest theorem. If the operator $\hat{A}$ is time independent, then

$$\frac{d}{dt}\langle A\rangle = \frac{1}{i\hbar}\langle[\hat{A}, H]\rangle$$

(a) Show that for a particle in a general potential $V(\vec{r})$, the rate of change of the expectation value of the orbital angular momentum $\vec{L}$ is equal to the expectation value of the torque $\vec{N} = \vec{r} \times (-\nabla V)$,

$$\frac{d}{dt}\langle\vec{L}\rangle = \langle\vec{N}\rangle$$

(b) Show that $d\langle\vec{L}\rangle/dt = 0$ for any spherically symmetric potential. This is the quantum analog of the classical conservation of angular momentum.