Physics 115B, Problem Set 2

Due Friday, April 15, 5pm

Every problem is worth 10 points. Every sub-question is worth the same, unless otherwise specified.

1 Radial Momentum

In lecture, we argued for an appropriate definition of the operator corresponding to momentum in the radial direction,

$$p_r \equiv \frac{1}{2}\left(\hat{r}\cdot\vec{p}+\vec{p}\cdot\hat{r}\right) = -\frac{i\hbar}{2}\left(\frac{1}{r}\vec{r}\cdot\nabla+\nabla\cdot(\vec{r}/r)\right)$$

(a) Compute $\hat{r}\cdot\vec{p}-\vec{p}\cdot\hat{r}$. Use your result to show that $\hat{r}\cdot\vec{p}$ is not Hermitian, and hence cannot itself be identified with momentum in the radial direction.

(b) Now using the definition of p_r given above, show that

$$p_r^2 = -\frac{\hbar^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$$

2 Algebraic Harmonic Oscillator

In lecture we noted that the 3d harmonic oscillator could be solved in spherical coordinates with algebraic techniques, where the effective radial Hamiltonian H_ℓ acting on a radial wavefunction R_ℓ could be expressed in terms of the lowering operator

$$a_\ell \equiv \frac{1}{\sqrt{2m\hbar\omega}}\left[ip_r - \frac{(\ell+1)\hbar}{r} + m\omega r\right]$$

(a) Show explicitly that $[a_\ell, a_\ell^\dagger] = \frac{(\ell+1)\hbar}{m\omega r^2} + 1$ (4 points)

(b) Show explicitly that $H_\ell = \hbar\omega\left(a_\ell^\dagger a_\ell + (\ell+3/2)\right)$ (3 points)

(c) Show explicitly that $a_\ell^\dagger$ is a raising operator that takes a state of given total energy E_n and fixed ℓ into a state of higher total energy $E_n+\hbar\omega$ with $\ell\to\ell-1$. (3 points)

3 Algebraic Free Particle

(every sub-question worth 2 points) Consider a stationary state $\psi_{\ell,m}(r,\theta,\phi)$ of energy E and fixed ℓ, m describing a free particle of mass M in three dimensions in spherical coordinates.

(a) Starting from the time-independent Schrödinger equation satisifed by ψ_ℓ, show that the radial part of the wavefunction $R_\ell(r)$ satisfies $H_\ell R_\ell = E R_\ell$, where

$$H_\ell \equiv \frac{1}{2M}\left(p_r^2 + \frac{\ell(\ell+1)\hbar^2}{r^2}\right)$$

What is the physical interpretation of H_ℓ in this case?

(b) Show that you can write $H_\ell = a_\ell^\dagger a_\ell$, where now

$$a_\ell \equiv \frac{1}{\sqrt{2M}}\left(ip_r - \frac{(\ell+1)\hbar}{r}\right)$$

(c) Show that $[a_\ell, a_\ell^\dagger] = H_{\ell+1} - H_\ell$

(d) Compute $[a_\ell, H_\ell]$. What is the state $a_\ell R_\ell$?

(e) Show that (in contrast to the harmonic oscillator) for any fixed $E > 0$ there is no upper bound on ℓ. Interpret this physically.

4 Good Coordinates for the Hydrogen Atom

The Hamiltonian describing an electron, a proton, and their electrostatic binding energy (collectively, the hydrogen atom) in position space is

$$H = -\frac{\hbar^2}{2m_p}\nabla_p^2 - \frac{\hbar^2}{2m_e}\nabla_e^2 - \frac{e^2}{4\pi\epsilon_0|\vec{x}_e - \vec{x}_p|}$$

where the subscripts p, e respectively denote the proton and the electron, and e.g. ∇_p^2 implies the use of derivatives with respect to the components of $\vec{x}_p$. This depends on six variables, but can be reduced in analogy with the classical 2-body problem to one that depends only on the relative separation of the two particles. Our starting point is to define the center-of-mass coordinate $\vec{X}$ and the relative coordinate $\vec{r}$, respectively given by

$$\vec{X} \equiv \frac{m_e\vec{x}_e + m_p\vec{x}_p}{m_e + m_p} \qquad \vec{r} \equiv \vec{x}_e - \vec{x}_p$$

(a) Show that

$$\nabla_p^2 = \left(\frac{m_p}{m_e + m_p}\right)^2 \nabla_{\vec{X}}^2 + \nabla_{\vec{r}}^2 - \frac{2m_p}{m_e + m_p}\frac{\partial^2}{\partial\vec{X}\cdot\partial\vec{r}}$$

and determine the analogous expression for ∇_e^2. (4 points)

(b) Using your results from part (a), show that the time-independent Schrödinger equation can be written as

$$-\frac{\hbar^2}{2(m_e + m_p)}\nabla_{\vec{X}}^2\psi - \frac{\hbar^2}{2\mu}\nabla_{\vec{r}}^2\psi - \frac{e^2}{4\pi\epsilon_0 r}\psi = E\psi$$

(3 points)

(c) In general, the state ψ is a function of both $\vec{x}_p$ and $\vec{x}_e$, or equivalently of both $\vec{X}$ and $\vec{r}$. Use separation of variables in $\vec{X}$ and $\vec{r}$ to split your TISE from part (b) into two separate time-independent Schrödinger equations, one involving only the center-of-mass coordinate $\vec{X}$ and one involving only the relative coordinate $\vec{r}$. From this, show that the important physics of the hydrogen atom is contained in the solutions to

$$-\frac{\hbar^2}{2\mu}\nabla_{\vec{r}}^2\psi(\vec{r}) - \frac{e^2}{4\pi\epsilon_0 r}\psi(\vec{r}) = E_{\vec{r}}\psi(\vec{r})$$

where μ is the reduced mass and $E_{\vec{r}}$ is the energy associated with the relative coordinate, $E_{\vec{r}} \leq E$. (3 points)

5 The Size of Hydrogen

(a) Find $\langle r\rangle$ and $\langle r^2\rangle$ for an electron in the ground state of hydrogen. Express your answer in terms of the Bohr radius. (3 points)

(b) Find $\langle r\rangle$ and $\langle r^2\rangle$ for an electron in a circular orbit of hydrogen with arbitrary principal quantum number n (this corresponds to $\ell = n - 1$ and any of the allowed m). (3 points)

(c) Compute the RMS uncertainty $\sqrt{\langle r^2\rangle - \langle r\rangle^2}$ in r for the electron in part (b). (2 points)

(d) Using your answer from part (b), how much more volume does a hydrogen atom in the $n = 100$ state occupy compared to a hydrogen atom in the ground state? (2 points)

6 Hydrogenic Atoms

A *hydrogenic atom* consists of a single electron orbiting a nucleus with Z protons. Determine the energies $E_n(Z)$, the binding energy $E_1(Z)$, the Bohr radius $a_0(Z)$, and the Rydberg constant $\mathcal{R}(Z)$ for these atoms, in terms of multiples of the corresponding answers for hydrogen. There's nothing much to calculate here – Z appears multiplying the potential energy in the Hamiltonian, and the rest amounts to following through the Z dependence appropriately.