1. Inside the box \( -\frac{\hbar^2}{2m} \nabla^2 \psi = E\psi \)

\[ -\frac{\hbar}{2m} \left[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right] = E\psi \quad (1) \]

Separation of variables \( \psi(x,y,z) = X(x)Y(y)Z(z) \)

Substituting into (1), dividing by \( \psi \), multiplying by \( -\frac{2m}{\hbar} \):

\[ \frac{1}{X} \frac{d^2X}{dx^2} + \frac{1}{Y} \frac{d^2Y}{dy^2} + \frac{1}{Z} \frac{d^2Z}{dz^2} = -\frac{2mE}{\hbar} \]

Each individual term must be equal to a constant.

Let's write \( \frac{1}{X} \frac{d^2X}{dx^2} = -k_x^2 \), \( \frac{1}{Y} \frac{d^2Y}{dy^2} = -k_y^2 \), \( \frac{1}{Z} \frac{d^2Z}{dz^2} = -k_z^2 \)

with \( k_x^2 + k_y^2 + k_z^2 = \frac{2mE}{\hbar} \) \( \Rightarrow E = \frac{\hbar^2}{2m} (k_x^2 + k_y^2 + k_z^2) \)

The solution of the \( X \) equation is

\( X(x) = A_x \sin k_x x + B_x \cos k_x x \)
The boundary conditions are \( X(0) = X(L) = 0 \).

This gives \( B_x = 0 \) and \( k_x = \frac{n_x \pi}{a} \), \( n_x \) is integer.

Similarly for the \( y \) and \( z \) equations.

Solution then is \( \psi(x, y, z) = C \sin \frac{n_x \pi}{a} \sin \frac{n_y \pi}{a} \sin \frac{n_z \pi}{a} \).

Normalization of the wavefunction \( \int_0^a \int_0^a \int_0^a |\psi|^2 \, dx \, dy \, dz = 1 \) gives \( C = (\frac{2}{a})^{3/2} \).

\[ E = \frac{\hbar^2 \pi^2}{2m a^2} (n_x^2 + n_y^2 + n_z^2) \]

(b) Lowest energy \( n \)'s = \( 1 \) \( E_1 = \frac{3 \hbar^2 \pi^2}{2m a^2} \) only 1 combination.

Next lowest \( (n_x, n_y, n_z) = (2, 1, 1) \) and 3 permutations \( E_2 = \frac{6 \pi^2 \hbar^2}{2m a^2} \) degeneracy = 3

3rd lowest \( (n_x, n_y, n_z) = (2, 2, 1) \) and 3 permutations \( E_3 = \frac{9 \pi^2 \hbar^2}{2m a^2} \) degeneracy = 3

4th lowest \( (n_x, n_y, n_z) = (3, 1, 1) \) and 3 permutations \( E_4 = \frac{14 \pi^2 \hbar^2}{2m a^2} \) degeneracy = 3

5th lowest \( (n_x, n_y, n_z) = (2, 2, 2) \) \( E_5 = \frac{12 \pi^2 \hbar^2}{2m a^2} \) degeneracy = 1

6th lowest \( (n_x, n_y, n_z) = (1, 2, 3) \) 6 permutations \( E_6 = \frac{14 \pi^2 \hbar^2}{2m a^2} \) degeneracy = 6
\( -\frac{\hbar^2}{2m} \nabla^2 \Psi + \frac{1}{2} m \omega^2 (x^2 + y^2 + z^2) \Psi = E \Psi \)

Using the same method as in problem 1, we obtain 3 different equations that look exactly like 1D oscillators, e.g.

\[ -\frac{\hbar^2}{2m} \frac{d^2 x}{d x^2} + \frac{1}{2} m \omega^2 x^2 X(x) = E_x X(x) \]

This has solutions (from last quarter) \( E_x = (n_x + \frac{1}{2}) \hbar \omega \)

Same for the \( y \) and \( z \) coordinates, and

\[ E = E_x + E_y + E_z = (n_x + n_y + n_z + \frac{3}{2}) \hbar \omega = (n + \frac{3}{2}) \hbar \omega \]

Note: in this case the \( n_x, n_y, n_z \) can be \( = 0 \)

(b) The degeneracy of the state with eigenenergy \( E_n = (n + \frac{3}{2}) \hbar \omega \) is equal to the number of ways that 3 integers can add up to \( n \). For a given \( n_x \), in order to have \( n_x + n_y + n_z = n \), I can pick \( n_y = 0, 1, \ldots, n-n_x \). (Note, once \( n_x \) and \( n_y \) are fixed, so is \( n_z = n-n_x-n_y \).) So, at a given \( n_x \) there are \( n-n_x+1 \) choices of \( n_y \). But, since \( n_x \) can be anything from 0 to \( n \), the degeneracy \( d(n) \) will be

\[ d(n) = \sum_{n_x=0}^{n} (n-n_x+1) = (n+1) \sum_{n_y=0}^{n} 1 - \sum_{n_y=0}^{n} n_x \]
\[ d(n) = (n+1)^2 - \frac{1}{2} n (n+1) = \frac{1}{2} (n+1)(n+2) = d(n) \]

(c) Any linear combination of degenerate eigenstates is an eigenstate. So \( \frac{1}{\sqrt{3}} [\psi_{100} + \psi_{010} + \psi_{001}] \) and \( \frac{1}{\sqrt{2}} [\psi_{000} + i\psi_{001}] \) are eigenstates, whereas \( \frac{1}{\sqrt{2}} [\psi_{000} + \psi_{001}] \) is not.

(3) Some of (2), except now

\[ E = (n_x + \frac{1}{2}) \hbar \omega_x + (n_y + \frac{1}{2}) \hbar \omega_y + (n_z + \frac{1}{2}) \hbar \omega_z \]

There is no degeneracy except in special cases.

(4) For \( l = 0 \), there is no angular dependence \( Y_0^0(\theta, \phi) = \sqrt{\frac{1}{4\pi}} \)

The radial equation becomes

\[ \frac{d^2 u}{dr^2} + k^2 u = 0 \]

with \( u(r) = rR(r) \) and \( k = \frac{\sqrt{2me}}{\hbar} \) [see equations 4.41 and 4.42 in Griffiths]

The solution is \( u(r) = Ae^{ikr} + Be^{-ikr} \)

or equivalently \( u(r) = C \sin kr + D \cos kr \)
Let's take the exponential solution

\[ R(r) = \frac{u(r)}{r} = A e^{i \kappa r} + B e^{-i \kappa r} \]

Note: This is a free particle - there is no way to normalize it - you should have seen this last quarter. See Griffiths page 56 - In analogy to the 1D case, the two solutions correspond to ingoing and outgoing waves (see Griffiths page 55).

(5) \[ P_c^m(z) = (1-z^2)^{m/2} F(z) \]

\[ \frac{d P_c^m}{dz} = -m z (1-z^2)^{m/2-1} F(z) + (1-z^2)^{m/2} \frac{dF(z)}{dz} \]

Then the associated Legendre equation in terms of \( F(z) \)

\[ \frac{d}{dz} \left[ (1-z^2) \left\{ \frac{d}{dz} \left[ -m z (1-z^2)^{m/2-1} F(z) + (1-z^2)^{m/2} \frac{dF(z)}{dz} \right] \right\} \right] + \]

\[ \left[ l(l+1) - \frac{m^2}{1-z^2} \right] (1-z^2)^{m/2} F(z) = 0 \]
\[
\frac{d}{dz} \left[-mz(1-z^2)^{M_2} F(z) + (1-z^2)^{M_2+1} \frac{dF}{dz}\right] + \\
+ l(l+1)(1-z^2)^{M_2} F(z) - m^2 (1-z^2)^{M_2-1} F(z) = 0
\]

\[
-m (1-z^2)^{M_2} F(z) + m^2 z^2 (1-z^2)^{M_2-1} F(z) - m z(1-z^2)^{M_2} \frac{dF}{dz} + \\
- 2z (\frac{M}{2} + 1) (1-z^2)^{M_2} \frac{dF}{dz} + (1-z^2)^{M_2+1} \frac{d^2F}{dz^2} + \\
+ l(l+1)(1-z^2)^{M_2} F(z) - m^2 (1-z^2)^{M_2-1} F(z) = 0
\]

Cancel out a factor of \((1-z^2)^{M_2}\)

\[
-m F(z) + \frac{m^2 z^2}{1-z^2} F(z) - m z \frac{dF}{dz} - m z \frac{dF}{dz} - 2z \frac{dF}{dz} + (1-z^2) \frac{d^2F}{dz^2} + \\
+ l(l+1) F(z) - \frac{m^2}{1-z^2} F(z) = 0
\]

\[
(1-z^2) \frac{d^2F}{dz^2} - 2z (l+1) \frac{dF}{dz} + [l(l+1) - m(m+1)] F(z) = 0 \tag{1}
\]

Now take Legendre equation

\[
\frac{d}{dz} \left[(1-z^2) \frac{d F}{dz}\right] + l(l+1) F(z) = 0
\]

\[
(1-z^2) \frac{d^2F}{dz^2} - 2z \frac{dF}{dz} + l(l+1) F(z) = 0
\]
Differentiate this equation m times. Look at the 3 terms, one by one.

First term: \((1 - z^2) \frac{d^{2m} P}{dz^{2m}} - 2m z \frac{d^{1+m} P}{dz^{1+m}} - 2m(m-1) \frac{d^{m} P}{dz^{m}}\)

Second term: \(-2z \frac{d^{1+m} P}{dz^{1+m}} - 2m \frac{d^{m} P}{dz^{m}}\)

Third term: \(\ell(\ell+1) \frac{d^{m} P}{dz^{m}}\)

Putting it together:

\[
(1 - z^2) \frac{d^{2m} P}{dz^{2m}} - 2(m+1) z \frac{d^{1+m} P}{dz^{1+m}} + \left[\ell(\ell+1) - m(m+1)\right] \frac{d^{m} P}{dz^{m}} = 0
\]

But this is the same as equation (1) for \(F(z)\)

So \(\frac{d^{m} P}{dz^{m}} = F(z) = (1 - z^2)^{-m/2} P_m^m(z)\)

And \(P_m^m(z) = (1 - z^2)^{m/2} \frac{d^{m} P}{dz^{m}}\) (The additional factor of \((-1)^m\) is conventional)

This was done for \(m > 0\). Because the associated Legendre equation depends on \(m^2\) not \(m\), this will also work for \(m < 0\) provided \(m \rightarrow |m|\).