# Physics 115B, Problem Set 1

Due Friday, April 8, 5pm

Upload to Gradescope (code V8VERZ).
**To help out with grading, please circle your final answers.**

## 1 The infinite cubical well (10 points)

Consider a particle of mass $m$ in a 3d infinite "cubical well" of length $a$ to a side, corresponding to the potential

$$V(x,y,z) = \begin{cases} 0 & x,y,z \text{ all between 0 and } a \\ \infty & \text{otherwise} \end{cases}$$

You can think of this as a particle in a box with infinitely thick walls.

(a) Use separation of variables in Cartesian coordinates to find the stationary states and the corresponding energies. (7 points).

(b) Call the distinct energies $E_1, E_2, E_3, \ldots$ in order of increasing energy. Find $E_1, E_2, E_3, E_4, E_5$, and $E_6$ and determine their degeneracies. (3 points).

## 2 The 3d harmonic oscillator (10 points)

Consider a particle of mass $m$ in a three-dimensional harmonic oscillator potential, corresponding to

$$V(r) = \frac{1}{2} m\omega^2 r^2$$

(a) Using separation of variables in Cartesian coordinates, show that this factorizes into a sum of three one-dimensional harmonic oscillators, and use your knowledge of the properties of the 1d harmonic oscillator to determine the allowed energies. (6 points).

(b) Determine the degeneracy $d(n)$ (i.e., the number of states with the same energy) of the $n$th energy eigenvalue $E_n$. (2 points).

(c) As you hopefully found in part (b), the ground state $\psi_0(x, y, z)$ is nondegenerate. However, the first excited energy eigenvalue $E_1$ is threefold degenerate, with eigenstates

$$\psi_{1,0,0}(x, y, z) = \tilde{\psi}_1(x)\tilde{\psi}_0(y)\tilde{\psi}_0(z)$$
$$\psi_{0,1,0}(x, y, z) = \tilde{\psi}_0(x)\tilde{\psi}_1(y)\tilde{\psi}_0(z)$$
$$\psi_{0,0,1}(x, y, z) = \tilde{\psi}_0(x)\tilde{\psi}_0(y)\tilde{\psi}_1(z)$$

where $\tilde{\psi}_i$ denotes an eigenstate of the 1d harmonic oscillator. Are any of the linear combinations $\frac{1}{\sqrt{3}}(\psi_{1,0,0} + \psi_{0,1,0} + \psi_{0,0,1})$, $\frac{1}{\sqrt{2}}(\psi_0 + \psi_{0,1,0})$, or $\frac{1}{\sqrt{2}}(\psi_{1,0,0} + i\psi_{0,0,1})$ energy eigenstates? In each case, why or why not? (2 points).

## 3 The anisotropic 3d harmonic oscillator (10 points)

Consider instead an anisotropic oscillator potential

$$V(x, y, z) = \frac{1}{2}m\omega_x^2 x^2 + \frac{1}{2}m\omega_y^2 y^2 + \frac{1}{2}m\omega_z^2 z^2$$

where $\omega_x \neq \omega_y \neq \omega_z$. Using the same approach as in Problem (1), find the allowed energies. (8 points).
What happened to the degeneracies? Why? (2 points).

## 4 The free particle (10 points)

Consider a free particle in three dimensions. This may be viewed as a special case of rotationally invariant potentials. As such, write down the time-independent Schrödinger equation for a free particle in spherical coordinates. Solve this to find the wavefunction of a particle with energy eigenvalue $E$ for the special case of $\ell = 0$, i.e. when the separation constant between $r$ and $\theta, \phi$ vanishes.

## 5 Associated Legendre Polynomials (10 points)

Legendre polynomials $P_\ell(z)$ are the solutions to the ordinary Legendre equation

$$\frac{d}{dz}\left[(1 - z^2)\frac{dP_\ell}{dz}\right] + \ell(\ell + 1)P_\ell(z) = 0$$

Show that the associated Legendre polynomials $P_\ell^m(z)$ given by

$$P_\ell^m(z) = (-1)^m(1 - z^2)^{|m|/2}\left(\frac{d}{dz}\right)^{|m|} P_\ell(z)$$

are the solution of the associated Legendre equation

$$\frac{d}{dz}\left[(1-z^2)\frac{dP_\ell^m}{dz}\right]+\left[\ell(\ell+1)-\frac{m^2}{1-z^2}\right]P_\ell^m(z)=0$$

**Hint:** define $F(z)=(1-z^2)^{-m/2}P_\ell^m(z)$ for $m>0$ and rewrite the equation above in terms of $F(z)$. (5 points).

Next, take the ordinary Legendre equation for $P_\ell(z)$ and differentiate it $m$ times with respect to $z$. (3 points for correct differentiation, 2 points for drawing conclusions).

Remember Leibniz rule for differentiating a product of functions $n$ times.

$$(uv)^{(n)}=\sum_{i=0}^{n}\binom{n}{i}u^{(n-i)}v^{(i)}$$