Cross Products: $(\vec{a}\times\vec{b})_k = \epsilon_{ijk}a_i b_j$.

Helpful Integrals: $\int d^3x = \int_0^\infty r^2 dr \int d\Omega$, $\int d\Omega = \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi = 4\pi$, $\int_0^\infty r^n e^{-r/a}\, dr = n!\, a^{1+n}$.

Schrödinger Equation: $i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \hat{H}|\Psi\rangle$ w/ Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + V$

Harmonic oscillator (1d): For $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$, raising/lowering operators $a_\pm = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} \mp i\hat{p})$, $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$, $\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$, $[a_-, a_+] = 1$, $\hat{H} = \hbar\omega(a_+a_- + 1/2)$, $E_n = \hbar\omega(n+1/2)$, $a_+\psi_n = \sqrt{n+1}\psi_{n+1}$, $a_-\psi_n = \sqrt{n}\psi_{n-1}$, $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$.

Laplacian: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ (Cartesian), $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}\right)$ (spherical).

QM in 3D: Position operator $\vec{x} = (x,y,z)$ and momentum operator $\vec{p} = (p_x, p_y, p_z)$ in Cartesian coords. Position space $p_x = -i\hbar\frac{\partial}{\partial x}, p_y = -i\hbar\frac{\partial}{\partial y}, p_z = -i\hbar\frac{\partial}{\partial z}$, so $\vec{p} = -i\hbar\nabla$ and $H = -\frac{\hbar^2}{2m}\nabla^2 + V$. Commutators $[x,p_x] = [y,p_y] = [z,p_z] = i\hbar$, all other commutators of x, y, z, p_x, p_y, p_z are zero.

Spherically symmetric potentials: $V(\vec{r}) = V(r)$, eigenstates $\psi_{n,\ell,m} = R_{n,\ell}(r)Y_\ell^m(\theta,\phi)$, radial momentum $\hat{p}_r = -i\hbar\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)$ and $\hat{p}_r^2 = -\hbar^2\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$.

Radial equation: $\hat{H}_\ell R_\ell(r) = \left[\frac{\hat{p}_r^2}{2m} + V(r) + \frac{\hbar^2\ell(\ell+1)}{2mr^2}\right]R_\ell(r) = E_\ell R_\ell(r)$.

Harmonic oscillator (3d): $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{r}^2$. Operator $\hat{a}_\ell = \frac{1}{\sqrt{2m\hbar\omega}}(i\hat{p}_r + m\omega r - \frac{(\ell+1)\hbar}{r})$; $\hat{H}_\ell = \hbar\omega(\hat{a}_\ell^\dagger\hat{a}_\ell + l + 3/2)$. $\hat{a}_\ell$ raises ℓ by one and lowers E_ℓ by $\hbar\omega$ while $\hat{a}_\ell^\dagger$ does the opposite. $[\hat{a}_\ell, \hat{a}_\ell^\dagger] = \frac{\hat{H}_{\ell+1} - \hat{H}_\ell}{\hbar\omega} + 1$.

Spherical Harmonics: $Y_\ell^m(\theta,\phi)$ orthogonal in both ℓ, m. Simplest harmonics:

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2} \qquad Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \qquad Y_1^{\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta\, e^{\pm i\phi}$$

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Hydrogen atom: $V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$, energies $E_n = -\frac{\hbar^2}{2ma_0^2n^2} = -\mathcal{R}/n^2$, Rydberg constant $\mathcal{R} = 13.6$ eV, Bohr radius $a_0 \equiv \frac{4\pi\epsilon_0\hbar^2}{me^2} = 0.53$ Å, ground state wavefunction $\psi_{1,0,0}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}$, Operator $\hat{a}_\ell = \frac{a_0}{\sqrt{2}\hbar}(i\hat{p}_r + \frac{\hbar}{a_0(\ell+1)} - \frac{(\ell+1)\hbar}{r})$; $\hat{H}_\ell = \frac{\hbar^2}{ma_0^2}(\hat{a}_\ell^\dagger\hat{a}_\ell - \frac{1}{2(\ell+1)^2})$. $\hat{a}_\ell$ raises ℓ by one while $\hat{a}_\ell^\dagger$ does the opposite. $[\hat{a}_\ell, \hat{a}_\ell^\dagger] = \frac{ma_0^2}{\hbar^2}(\hat{H}_{\ell+1} - \hat{H}_\ell)$.

Angular Momentum: $[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k$. $\hat{L}^2|\ell,m\rangle = \hbar^2\ell(\ell+1)|\ell,m\rangle$, $\hat{L}_z|\ell,m\rangle = \hbar m|\ell,m\rangle$.

Raising and lowering operators: $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$ for eigenstates of $\hat{L}^2$ and $\hat{L}_z$.
Then: $\hat{L}_+|\ell,m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m+1)}\,|\ell,m+1\rangle$ and $\hat{L}_-|\ell,m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m-1)}\,|\ell,m-1\rangle$. In terms of these operators, $\hat{L}_x = \frac{1}{2}(\hat{L}_+ + \hat{L}_-)$ and $\hat{L}_y = \frac{1}{2i}(\hat{L}_+ - \hat{L}_-)$.

Matrix operators: $M_{ij} = \left\langle u_i\middle|\widehat{M}|u_j\right\rangle$ where the $|u_k\rangle$'s are the basis vectors.

Spin operators:

$$\widehat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \widehat{S}_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \qquad \widehat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

The three matrices above are the "Pauli matrices": σ_x, σ_y, and σ_z.

Representation of spin-1/2: Basis $\chi_+ = \begin{pmatrix}1\\0\end{pmatrix}$, $\chi_- = \begin{pmatrix}0\\1\end{pmatrix}$, operators $\vec{S} = (\hbar/2)\vec{\sigma}$

Isospin operators: $I_i = \frac{1}{2}\sigma_i$ where $i = 1, 2, 3$ or equivalently $i = x, y, z$.

Isospin doublets for nucleons, quarks, antiquarks:

$$\begin{pmatrix} p \\ n \end{pmatrix} \quad \begin{pmatrix} u \\ d \end{pmatrix} \quad \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}$$

Coupling to electromagnetism: For a particle of charge q, $H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + q\varphi$, where $\vec{p}$ is the vector of momentum operators (i.e. $\vec{p} = (p_x, p_y, p_z)$), $\vec{A}$ is the vector potential, φ is the scalar potential, in terms of which the electric and magnetic fields are $\vec{E} = -\nabla\varphi - \partial\vec{A}/\partial t$ and $\vec{B} = \nabla \times \vec{A}$. Gauge transformations $\varphi' \equiv \varphi - \partial\Lambda/\partial t$, $\vec{A}' \equiv \vec{A} + \nabla\Lambda$ leave the physics unchanged.

Free electron gas: Box of volume $V = L_x L_y L_z$. States labeled by wavenumbers $\vec{k} = (n_x\pi/L_x, n_y\pi/L_y, n_z\pi/L_z)$, each occupies k-space volume π^3/V. Electrons in ground state fill out states to Fermi radius $k_F = (3\rho\pi^2)^{1/3}$ in terms of free electron density $\rho = Nd/V$. Energy of the Fermi surface $E_F = \frac{\hbar^2 k_F^2}{2m}$, total energy $E_{tot} = \frac{\hbar^2 k_F^5 V}{10\pi^2 m} = \frac{\hbar^2 (3\pi^2 Nd)^{5/3}}{10\pi^2 m} V^{-2/3}$. Work $dW = PdV$ done by pressure implies degeneracy pressure $P = \frac{2}{3}\frac{E_{tot}}{V}$.

Symmetries and Conservation Laws: Active transformation: act on states, $|\psi\rangle \to T|\psi\rangle$, operators fixed. Passive transformation: act on operators, $\mathcal{O} \to T^\dagger \mathcal{O} T$, states fixed. Symmetry if $[H, \mathcal{O}] = 0$, implies conservation laws $\frac{d}{dt}\langle\mathcal{O}\rangle = 0$.

Transformations: Transformations unitary, $T^\dagger T = 1$. Translation operator $T(a)|x\rangle = |x + a\rangle$, $T(a)\psi(x) = \psi(x - a)$, generated by momentum, $T(a) = \exp[-ia\hat{p}/\hbar]$. Parity operator $\Pi|x\rangle = |-x\rangle$, $\Pi\psi(x) = \psi(-x)$. Rotation operator $R(\hat{n}, \theta)$ by θ around axis $\hat{n}$, generated by angular momentum, $R(\hat{n}, \theta) = \exp\left[-i\theta\hat{n}\cdot\vec{L}/\hbar\right]$. Time translation operator $U(t)$, generated by Hamiltonian, $U(t) = \exp[-iHt/\hbar]$. Eigenvalues of parity: ± 1. Continous symmetries: additive conservation laws; Discrete symmetries: multiplicative conservation laws.

Clebsch-Gordan: