PHYSICS 110 B
HOMEWORK 4

GRIFFITHS 9.16

$$A e^{iax} + B e^{ibx} = C e^{icx}$$

Set $x=0$ $\boxed{A+B=C}$

Multiply by e^{-iax} $\quad A + B e^{i(b-a)x} = C e^{i(c-a)x} \quad (1)$

Take imaginary part of (1)

$$B \sin(b-a)x = C \sin(c-a)x$$

Clearly this works if $a=b=c$. Now assume that this is not true and see if we can get to a contradiction (when a, b, c are non-zero)

If $b \neq a$ I can set $x = \frac{\pi}{b-a}$ which gives $0 = C \sin \frac{c-a}{b-a}\pi$

For $C \neq 0$, I must have $\frac{c-a}{b-a} = N$ where N = integer

Similarly if $c \neq a$ I can set $x = \frac{\pi}{c-a}$ which gives $B \sin \frac{b-a}{c-a}\pi = 0$

For $B \neq 0$, I must have $\frac{b-a}{c-a} = M$ where M = integer

The only way that the two circled equations can work is if $N = M = 1$

This means $b - a = c - a$ which gives $b = c$

So I can still have in principle $b = c \neq a$

Equation (1) then becomes (setting $c=b$)

$$A + Be^{i(b-a)x} = Ce^{i(b-a)x}$$

$$Ae^{i(a-b)x} = C - B$$

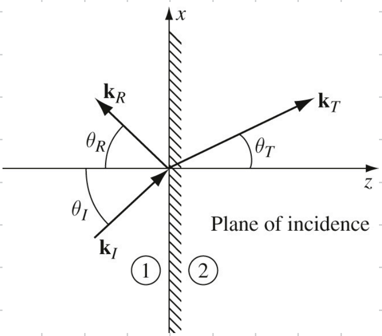
The imaginary part is $A\sin(a-b)x = 0$

In order for this to work at all values of x, I must have that $a=b$ and since $b=c$, $a=b=c$

$$\boxed{\text{CONTRADDICTION REACHED}}$$

GRIFFITHS 9.17

Will use the sketch from Figure 9.14 in Griffiths



And write the $\vec{k}$ vectors according to hep.ucsb.edu/people/claudio/ph110-W25/EM-polarizations.pdf

$$\hat{k}_I = \sin\theta_I \hat{x} + \cos\theta_I \hat{z}$$
$$\hat{k}_R = \sin\theta_R \hat{x} - \cos\theta_R \hat{z}$$
$$\hat{k}_T = \sin\theta_T \hat{x} + \cos\theta_T \hat{z}$$

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Dropping the $e^{i\vec{k}\cdot\vec{r}-\omega t}$ terms, on the surface we have

$$\vec{E}_I = E_{0_I}\hat{y} \qquad \vec{B}_I = \frac{1}{v_1}\left(\hat{k}_I \times \vec{E}_I\right) = \frac{E_{0_I}}{v_1}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \sin\theta_I & 0 & \cos\theta_I \\ 0 & 1 & 0 \end{vmatrix} = \frac{E_{0_I}}{v_1}\left[-\cos\theta_I\hat{x} + \sin\theta_I\hat{z}\right]$$

$$\vec{E}_R = E_{0_R}\hat{y} \longrightarrow \vec{B}_R = \frac{1}{v_1}\left(\hat{k}_R \times \vec{E}_R\right) = \frac{E_{0_R}}{v_1}\left[\cos\theta_R\hat{x} + \sin\theta_R\hat{z}\right]$$

$$\vec{E}_T = E_{0_T}\hat{y} \longrightarrow \vec{B}_T = \frac{E_{0_T}}{v_2}\left[-\cos\theta_T\hat{x} + \sin\theta_T\hat{z}\right]$$

Boundary conditions

$$\left.\begin{array}{ll}
\text{(i)} & \epsilon_1(\tilde{\mathbf{E}}_{0_I} + \tilde{\mathbf{E}}_{0_R})_z = \epsilon_2(\tilde{\mathbf{E}}_{0_T})_z, \\
\text{(ii)} & (\tilde{\mathbf{B}}_{0_I} + \tilde{\mathbf{B}}_{0_R})_z = (\tilde{\mathbf{B}}_{0_T})_z, \\
\text{(iii)} & (\tilde{\mathbf{E}}_{0_I} + \tilde{\mathbf{E}}_{0_R})_{x,y} = (\tilde{\mathbf{E}}_{0_T})_{x,y}, \\
\text{(iv)} & \frac{1}{\mu_1}(\tilde{\mathbf{B}}_{0_I} + \tilde{\mathbf{B}}_{0_R})_{x,y} = \frac{1}{\mu_2}(\tilde{\mathbf{B}}_{0_T})_{x,y},
\end{array}\right\}$$

From (i) we get nothing

From (ii) we get $\frac{1}{v_1}E_{0_I}\sin\theta_I + \frac{1}{v_1}E_{0_R}\sin\theta_R = \frac{1}{v_2}E_{0_T}\sin\theta_T$

Since $\theta_I = \theta_R$ this becomes $E_{0_I} + E_{0_R} = \frac{v_1}{v_2}\frac{\sin\theta_T}{\sin\theta_I}E_{0_T}$

But Snell's law $\frac{\sin\theta_T}{\sin\theta_I} = \frac{v_2}{v_1} \Rightarrow E_{0_I} + E_{0_R} = E_{0_T}$

From (iii) we get that only the y-component is involved and the equation that we get is exactly the same as the previous one

From (iv) we get $\frac{1}{\mu_1 v_1}\left[-\cos\theta_I E_{0_I} + \cos\theta_I E_{0_R}\right] = -\frac{E_{0_T}}{\mu_2 v_2}$

$$E_{0_I} - E_{0_R} = \frac{\mu_1 v_1 \cos\theta_T}{\mu_2 v_2 \cos\theta_I}E_{0_T}$$

Using $\alpha = \frac{\cos\theta_T}{\cos\theta_I}$ $\beta = \frac{\mu_1 v_1}{\mu_2 v_2}$ $E_{0I} - E_{0R} = \alpha\beta E_{0T}$

The sum of the two circled equations gives

$$2E_{0I} = (\alpha\beta + 1)E_{0T} \qquad \boxed{E_{0T} = \frac{2}{\alpha\beta+1} E_{0I}}$$

Substituting into the 1st circled equation

$$E_{0I} + E_{0R} = \frac{2}{\alpha\beta+1} E_{0I} \qquad E_{0R} = \frac{2-\alpha\beta-1}{\alpha\beta+1} E_{0I} \qquad \boxed{E_{0R} = \frac{1-\alpha\beta}{1+\alpha\beta} E_{0I}}$$

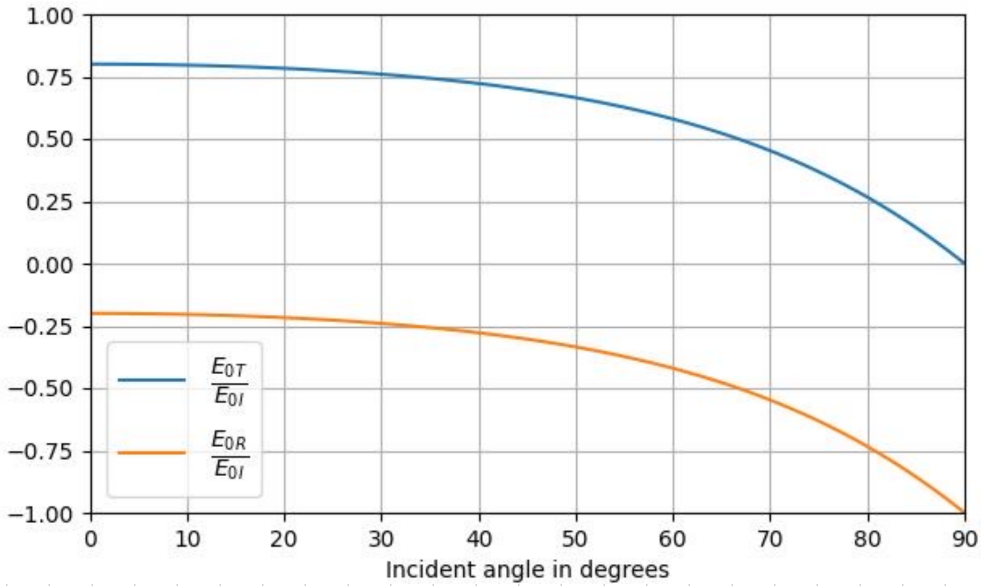
For the required sketch, we need $\alpha\beta$ _ We are also going to use $\mu_1 = \mu_2$

Equation 9.111 in Griffiths

$$\alpha = \frac{\sqrt{1-(n_1/n_2)^2 \sin^2\theta_I}}{\cos\theta_I} = \frac{\sqrt{1 - \sin^2\theta_I / \beta^2}}{\cos\theta_I}$$

$$\alpha\beta = \frac{\beta\sqrt{1-\sin^2\theta_I/\beta^2}}{\cos\theta_I} \therefore = \frac{\sqrt{\beta^2 - \sin^2\theta_I}}{\cos\theta_I}$$

Here then are the plots as requested (on the next page)



For a Brewster angle we want $E_{0R} = 0$ which means $\alpha\beta = 1$

We had $\alpha\beta = \frac{\sqrt{\beta^2 - \sin^2\theta_I}}{\cos\theta_I}$ which means $\cos\theta_I = \sqrt{\beta^2 - \sin^2\theta_I}$

This only works if $\beta = 1$, ie, if the two media have the same properties (optically indistinguishable)

At normal incidence $\alpha\beta = \beta$

$\boxed{E_{0T} = \frac{2}{1+\beta} E_{0I}}$ $\boxed{E_{0R} = \frac{1-\beta}{1+\beta} E_{0I}}$

Same as equation 9.83

For the reflection and transmission coefficient we use equations 9.116 and 9.117

$$R = \left(\frac{E_{0R}}{E_{0I}}\right)^2 \qquad R = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2$$

$$T = \alpha\beta \left(\frac{E_{0T}}{E_{0I}}\right)^2 = \frac{4\alpha\beta}{(\alpha\beta+1)^2}$$

$$R+T = \frac{1+\alpha^2\beta^2 - 2\alpha\beta + 4\alpha\beta}{(1+\alpha\beta)^2}$$

$$\boxed{R+T=1}$$

GRIFFITHS 9.21

(a) Eq 9.128

$$K = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1\right]^{1/2}$$

For $\sigma \ll \varepsilon\omega$ $\quad \sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} \simeq 1 + \frac{1}{2}\left(\frac{\sigma}{\varepsilon\omega}\right)^2$

Thus $\quad K = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\frac{1}{2}\left(\frac{\sigma}{\varepsilon\omega}\right)^2\right] = \frac{\omega}{2}\sqrt{\varepsilon\mu}\,\frac{\sigma}{\varepsilon\omega}$

$$K = \frac{1}{2}\sigma\sqrt{\frac{\mu}{\varepsilon}} \qquad d = 1/K \implies \boxed{d = \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}}$$

For H_2O $\quad \varepsilon \cong 80\,\varepsilon_0$ (From Wikipedia, at 20°C)

$\mu \simeq \mu_0$ (not magnetic)

$\sigma \simeq 5.5\ 10^{-6}\,S/m$ (From google search, ultra pure water)

$$d = \frac{2}{5.5\ 10^{-6}} \sqrt{\frac{80 \cdot 9\ 10^{-12}}{4\pi\ 10^{-7}}}\ m \cong 3.6\ 10^{5} \sqrt{5.7\ 10^{-4}}$$

$$\boxed{d \simeq 28\,Km}$$

(b) if $\sigma \gg \varepsilon\omega$ then looking at equation 9.128 we see that $k \cong K$

So $d = \frac{1}{k}$ and since $k = \frac{2\pi}{\lambda}$. $\boxed{d = \frac{\lambda}{2\pi}}$

In terms of ω, ε, etc:

$$k \cong \omega \sqrt{\frac{\varepsilon\mu}{2}} \left[\sqrt{\left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1 \right]^{1/2} \qquad \text{for } \sigma \gg \varepsilon\omega$$

$$k = \sqrt{\frac{\omega\mu\sigma}{2}} \qquad d = \sqrt{\frac{2}{\omega\mu\sigma}}$$

$$d = \sqrt{\frac{2}{10^{15}\ 4\pi\ 10^{-7}\ 10^{7}}}\ m \qquad \boxed{d \simeq 10\ nm}$$

Skin depth extremely short ⇒ OPAQUE

(c) Since $k \simeq K$ from equation 9.136

$$\boxed{\phi = \tan^{-1}\frac{K}{k} = \tan^{-1} 1 = 45°}$$

Equation 9.139 in the limit $\frac{\sigma}{\varepsilon\omega} \gg 1$

$$\frac{B_0}{E_0} \simeq \sqrt{\varepsilon\mu \frac{\sigma}{\varepsilon\omega}} = \sqrt{\frac{\mu\sigma}{\omega}}$$

For $\sigma \sim 10^{7}$ $\omega \sim 10^{15}$ (in SI units)

$$\frac{B_0}{E_0} = \sqrt{\frac{4\pi 10^{-7} \cdot 10^{7}}{10^{15}}}$$

$$\boxed{\frac{B_0}{E_0} \sim 10^{-7} \frac{s}{m}}$$

Compare in vacuum $\frac{B_0}{E_0} = \frac{1}{c}$

$$\boxed{\frac{B_0}{E_0} \sim 3\,10^{-9} \frac{s}{m}}$$