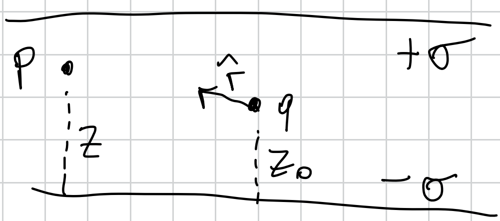
SESSION 2

Griffiths 8.2

(a) Total energy $U = \varepsilon_0 \vec{E}\cdot\vec{E}$

$$U = \frac{\varepsilon_0}{2}\left(E_\sigma^2 + E_q^2 + 2\vec{E}_\sigma\cdot\vec{E}_q\right)$$

Interaction energy $U_{INT} = \varepsilon_0 \vec{E}_\sigma\cdot\vec{E}_q$



The fields at P are $\vec{E}_\sigma = -\dfrac{\sigma}{\varepsilon_0}\hat{z}$

$$\vec{E}_q = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

$$\varepsilon_0 \vec{E}_\sigma\cdot\vec{E}_q = \frac{-\sigma q}{4\pi\varepsilon_0 r^2}\hat{r}\cdot\hat{z}$$

Take the origin at q

$$\hat{r} = (x\hat{x} + y\hat{y} + z\hat{z})\, r$$

$$\varepsilon_0 \vec{E}_\sigma \vec{E}_q = \frac{-\sigma q}{4\pi\varepsilon_0} \frac{z}{r^3}$$

Integrate over the volume of the capacitor

$$\int_V \varepsilon_0 \vec{E}_\sigma \vec{E}_q \, d\tau = \frac{-\sigma q}{4\pi\varepsilon_0} \int_{-z_0}^{d-z_0} dz \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{z}{(x^2+y^2+z^2)^{3/2}}$$

Use www.integral-calculator.com

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2+b^2)^{3/2}} = \frac{2}{a^2+b^2}$$

$$\int_V \varepsilon_0 \vec{E}_\sigma \vec{E}_q \, d\tau = -\frac{\sigma q}{2\pi\varepsilon_0} \int_{-z_0}^{d-z_0} dz \int_{-\infty} dy \frac{z}{y^2+z^2}$$

Some nice website

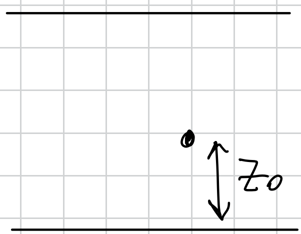
$$\int_{-\infty}^{\infty} \frac{dy}{y^2+a^2} = \frac{\pi}{|a|}$$

$$\int_V \varepsilon_0 \vec{E}_\sigma \vec{E}_q \, d\tau = -\frac{\sigma q}{2\varepsilon_0} \int_{-z}^{d-z_0} \frac{z}{|z|} \, dz$$

$$\int_V \varepsilon_0 \vec{E}_\sigma \vec{E}_q \, d\tau = -\frac{\sigma q}{2\varepsilon_0} \left[\int_{-z_0}^{0} (-1) \, dz + \int_{0}^{d-z_0} dz \right]$$

$$\boxed{\int_V \varepsilon_0 \vec{E}_\sigma \vec{E}_q \, d\tau = -\frac{\sigma q}{2\varepsilon_0} (d - 2z_0)}$$

(b) The potential energy is the work required to move q from the center to $z=0$



distance $\frac{d}{2} - z_0$

$$W = -qE_\sigma (d - 2z_0)$$

$$\boxed{W = -\frac{\sigma q}{2\varepsilon_0} (d - 2z_0)}$$

(c) $\vec{F} = -\vec{\nabla} W = -\frac{\partial}{\partial z_0} \left(-\frac{\sigma q}{2\varepsilon_0} (d - 2z_0) \right) \hat{z}$

$$\boxed{\vec{F} = -\frac{q\sigma}{\varepsilon_0}\hat{z} = q\vec{E}'_\sigma}$$