Strictly speaking the solution to equation (1)
is missing a piece

Eqtn (1) was

$$I_T = \frac{1}{w_0^2} \frac{d^2 I_L}{dt^2} + I_L$$

$$I_0 e^{iwt} = \frac{1}{w_0^2} \frac{d^2 I_L}{dt^2} + I_L$$

The full solution is

$$I_L = \frac{w_0^2}{w^2 - w_0^2} \cos wt + A \sin w_0 t + B \cos w_0 t$$

The A & B constants would be determined
by initial conditions that were not specified
in the statement of the problem