VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient: $\quad \nabla t = \dfrac{\partial t}{\partial x}\hat{\mathbf{x}} + \dfrac{\partial t}{\partial y}\hat{\mathbf{y}} + \dfrac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} + \dfrac{\partial v_z}{\partial z}$

Curl: $\quad \nabla \times \mathbf{v} = \left(\dfrac{\partial v_z}{\partial y} - \dfrac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\dfrac{\partial v_x}{\partial z} - \dfrac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\dfrac{\partial v_y}{\partial x} - \dfrac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian: $\quad \nabla^2 t = \dfrac{\partial^2 t}{\partial x^2} + \dfrac{\partial^2 t}{\partial y^2} + \dfrac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2 \sin\theta\,dr\,d\theta\,d\phi$

Gradient: $\quad \nabla t = \dfrac{\partial t}{\partial r}\hat{\mathbf{r}} + \dfrac{1}{r}\dfrac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \dfrac{1}{r\sin\theta}\dfrac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

Divergence: $\nabla \cdot \mathbf{v} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 v_r) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial v_\phi}{\partial \phi}$

Curl: $\quad \nabla \times \mathbf{v} = \dfrac{1}{r\sin\theta}\left[\dfrac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \dfrac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$

$$+ \dfrac{1}{r}\left[\dfrac{1}{\sin\theta}\dfrac{\partial v_r}{\partial \phi} - \dfrac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \dfrac{1}{r}\left[\dfrac{\partial}{\partial r}(r v_\theta) - \dfrac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $\quad \nabla^2 t = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial t}{\partial r}\right) + \dfrac{1}{r^2 \sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta \dfrac{\partial t}{\partial \theta}\right) + \dfrac{1}{r^2 \sin^2\theta}\dfrac{\partial^2 t}{\partial \phi^2}$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $\quad \nabla t = \dfrac{\partial t}{\partial s}\hat{\mathbf{s}} + \dfrac{1}{s}\dfrac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \dfrac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \dfrac{1}{s}\dfrac{\partial}{\partial s}(s v_s) + \dfrac{1}{s}\dfrac{\partial v_\phi}{\partial \phi} + \dfrac{\partial v_z}{\partial z}$

Curl: $\quad \nabla \times \mathbf{v} = \left[\dfrac{1}{s}\dfrac{\partial v_z}{\partial \phi} - \dfrac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\dfrac{\partial v_s}{\partial z} - \dfrac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \dfrac{1}{s}\left[\dfrac{\partial}{\partial s}(s v_\phi) - \dfrac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$

Laplacian: $\quad \nabla^2 t = \dfrac{1}{s}\dfrac{\partial}{\partial s}\left(s \dfrac{\partial t}{\partial s}\right) + \dfrac{1}{s^2}\dfrac{\partial^2 t}{\partial \phi^2} + \dfrac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\quad \int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\quad \int (\nabla \cdot \mathbf{A})\,d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\quad \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general:

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho \\[2mm] \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\[2mm] \nabla \cdot \mathbf{B} = 0 \\[2mm] \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter:

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\[2mm] \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\[2mm] \nabla \cdot \mathbf{B} = 0 \\[2mm] \nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions:

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\[2mm] \mathbf{H} = \dfrac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media:

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\[2mm] \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \dfrac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy:
$$U = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2 \right) d\tau$$

Momentum:
$$\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$$

Poynting vector:
$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$$

Larmor formula:
$$P = \frac{\mu_0}{6\pi c}q^2 a^2$$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2/Nm^2}$ (permittivity of free space)

$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}$ (permeability of free space)

$c = 3.00 \times 10^8\ \mathrm{m/s}$ (speed of light)

$e = 1.60 \times 10^{-19}\ \mathrm{C}$ (charge of the electron)

$m = 9.11 \times 10^{-31}\ \mathrm{kg}$ (mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$$\begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z \right) \\ \phi = \tan^{-1}(y/x) \end{cases}$$

$$\begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases}$$

$$\begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases}$$

$$\begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

Ohm's Law: $\mathbf{J} = \sigma \mathbf{E} = (1/\rho)\mathbf{E}$.   Continuity Equation: $\nabla \cdot \mathbf{J} = -\partial \rho / \partial t$

Mutual Inductance: $\text{emf}_2 = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt}$ Self Inductance: $\text{emf}_1 = -\frac{d\Phi_1}{dt} = -L\frac{dI_1}{dt}$

Maxwell Equations in integral form:

$$\int_S \mathbf{D} \cdot d\mathbf{a} = Q_{fenclosed} \qquad \int_S \mathbf{B} \cdot d\mathbf{a} = 0$$

$$\int_P \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{a} \qquad \int_P \mathbf{H} \cdot d\mathbf{l} = I_{enclosed} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{a}$$

Energy density: $u = \frac{1}{2}(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2)$ Poynting vector: $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Boundary conditions at the interface of two materials.

$$D_1^\perp - D_2^\perp = \sigma_f \qquad\qquad B_1^\perp - B_2^\perp = 0$$
$$\mathbf{E_1^\|} - \mathbf{E_2^\|} = 0 \qquad\qquad \mathbf{H_1^\|} - \mathbf{H_2^\|} = \mathbf{K}_f \times \hat{\mathbf{n}}$$

Wave equation in vacuum:

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

Speed of light in vacuum $c = 1/\sqrt{\epsilon_0 \mu_0}$.   In dielectric $v = \frac{c}{n}$ where $n = \sqrt{\epsilon \mu / \epsilon_0 \mu_0}$.

Plane wave solution propagating in direction $\hat{\mathbf{k}}$ (not in metals)

$$\mathbf{E} = \mathbf{E_0} \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta) \qquad \mathbf{B} = \frac{1}{v}\hat{\mathbf{k}} \times \mathbf{E}$$

with $\mathbf{E_0}$ perpendicular to $\hat{\mathbf{k}}$ and $v = \omega / k$.

Snell's law: $\frac{\sin \theta_T}{\sin \theta_I} = \frac{n_1}{n_2}$.

For linearly polarized waves with polarization in the material boundary plane (not in metals):

$$E_{0R} = \frac{\alpha - \beta}{\alpha + \beta}E_{0I} \qquad E_{0T} = \frac{2}{\alpha + \beta}E_{0I} \qquad \alpha = \frac{\cos \theta_T}{\cos \theta_I} \qquad \beta = \frac{\mu_1 v_1}{\mu_2 v_2}$$

Wave equation in metal:

$$\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{E}}{\partial t} \qquad\qquad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{B}}{\partial t}$$

Linearly polarized solution for propagation in z direction (metals):

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0 \, e^{-\kappa z} e^{i(kz - \omega t)} = \tilde{E}_0 \, e^{-\kappa z} e^{i(kz - \omega t)}\hat{x} \qquad\qquad \tilde{\mathbf{B}} = \tilde{\mathbf{B}}_0 \, e^{-\kappa z} e^{i(kz - \omega t)} = \frac{\tilde{k}}{\omega}\tilde{E}_0 \, e^{-\kappa z} e^{i(kz - \omega t)}\hat{y}$$

$$\tilde{k} = k + i\kappa \qquad k = \omega \sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1\right]^{1/2} \qquad \kappa = \omega \sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1\right]^{1/2}$$

Reflection from conductive surface at 90° incidence:

$$\tilde{\mathbf{E}}_{0R} = \frac{1 - \tilde{\beta}}{1 + \tilde{\beta}}\tilde{\mathbf{E}}_{0I} \qquad \tilde{\mathbf{E}}_{0T} = \frac{2}{1 + \tilde{\beta}}\tilde{\mathbf{E}}_{0I} \qquad \tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2$$