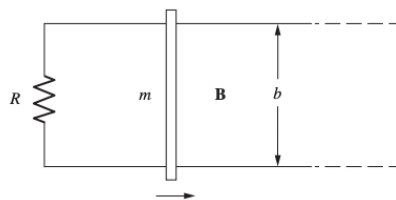


# Physics 110B, Winter 2025, Midterm Exam

Please put a “box” around each of your final answers.

## Problem 1

A metal bar of mass  $m$  slides without friction on two long parallel metal rails a distance  $b$  apart. A resistor  $R$  is connected across the rails at one end. The resistance of the bar and the rails is negligible. A uniform magnetic field  $\mathbf{B}$  is perpendicular to the plane of the figure. At time  $t = 0$  the bar is given a velocity  $v_0$  to the right. Find  $v(t)$ , the velocity of the bar as a function of time.



## Problem 2

An electric field in free space is

$$\mathbf{E} = E_0(\hat{\mathbf{x}} + \hat{\mathbf{y}}) \cos\left(\frac{2\pi}{\lambda}(z + ct)\right)$$

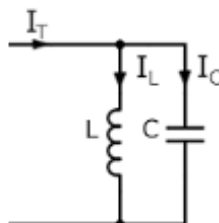
What is the magnetic field?

## Problem 3

Consider the propagation of linearly polarized wave in a **good** metal such as copper. Let the conductivity be  $\sigma$ , the angular frequency be  $\omega$ , the permittivity be  $\epsilon$ , and the permeability be  $\mu$ . Starting from one of the equations in the provided “cheat sheet”, derive an approximate expression for the skin depth.

## Problem 4

Consider the portion of circuit below. The current  $I_T$  is provided externally and is  $I_T = I_o \cos \omega t$ , or, in complex notation,  $I_T = \tilde{I}_0 e^{i\omega t}$ , where in this case  $\tilde{I}_0 = I_0$  is a real number. Find the current  $I_L$  as a function of time. You can work with complex voltages/currents or real voltages/currents, your choice. If you work with complex voltages/currents, for full credit you **do not** have to take the real part of your answer at the very end.



## VECTOR DERIVATIVES

**Cartesian.**  $d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}; \quad d\tau = dx dy dz$

*Gradient:*  $\nabla t = \frac{\partial t}{\partial x} \hat{\mathbf{x}} + \frac{\partial t}{\partial y} \hat{\mathbf{y}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$

*Divergence:*  $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

*Curl:*  $\nabla \times \mathbf{v} = \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{\mathbf{x}} + \left( \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{\mathbf{y}} + \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\mathbf{z}}$

*Laplacian:*  $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

**Spherical.**  $d\mathbf{l} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin \theta d\phi \hat{\boldsymbol{\phi}}; \quad d\tau = r^2 \sin \theta dr d\theta d\phi$

*Gradient:*  $\nabla t = \frac{\partial t}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial t}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial t}{\partial \phi} \hat{\boldsymbol{\phi}}$

*Divergence:*  $\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}$

*Curl:*  $\nabla \times \mathbf{v} = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{\mathbf{r}}$   
 $+ \frac{1}{r} \left[ \frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\boldsymbol{\theta}} + \frac{1}{r} \left[ \frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\boldsymbol{\phi}}$

*Laplacian:*  $\nabla^2 t = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial t}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial t}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 t}{\partial \phi^2}$

**Cylindrical.**  $d\mathbf{l} = ds \hat{\mathbf{s}} + s d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{z}}; \quad d\tau = s ds d\phi dz$

*Gradient:*  $\nabla t = \frac{\partial t}{\partial s} \hat{\mathbf{s}} + \frac{1}{s} \frac{\partial t}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$

*Divergence:*  $\nabla \cdot \mathbf{v} = \frac{1}{s} \frac{\partial}{\partial s} (s v_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

*Curl:*  $\nabla \times \mathbf{v} = \left[ \frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] \hat{\mathbf{s}} + \left[ \frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right] \hat{\boldsymbol{\phi}} + \frac{1}{s} \left[ \frac{\partial}{\partial s} (s v_\phi) - \frac{\partial v_s}{\partial \phi} \right] \hat{\mathbf{z}}$

*Laplacian:*  $\nabla^2 t = \frac{1}{s} \frac{\partial}{\partial s} \left( s \frac{\partial t}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$

## VECTOR IDENTITIES

## Triple Products

(1)  $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2)  $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

## Product Rules

(3)  $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4)  $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5)  $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6)  $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7)  $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8)  $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

## Second Derivatives

(9)  $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10)  $\nabla \times (\nabla f) = 0$

(11)  $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

## FUNDAMENTAL THEOREMS

**Gradient Theorem:**  $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

**Divergence Theorem:**  $\int (\nabla \cdot \mathbf{A}) d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

**Curl Theorem:**  $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

## BASIC EQUATIONS OF ELECTRODYNAMICS

## Maxwell's Equations

In general:

$$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter:

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{cases}$$

## Auxiliary Fields

Definitions:

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \end{cases}$$

Linear media:

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, & \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, & \mathbf{H} = \frac{1}{\mu} \mathbf{B} \end{cases}$$

## Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

## Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

## Energy, Momentum, and Power

$$\text{Energy:} \quad U = \frac{1}{2} \int (\epsilon_0 E^2 + \frac{1}{\mu_0} B^2) d\tau$$

$$\text{Momentum:} \quad \mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B}) d\tau$$

$$\text{Poynting vector:} \quad \mathbf{S} = \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B})$$

$$\text{Larmor formula:} \quad P = \frac{\mu_0}{6\pi c} q^2 a^2$$

## FUNDAMENTAL CONSTANTS

$$\epsilon_0 = 8.85 \times 10^{-12} \text{ C}^2/\text{Nm}^2 \quad (\text{permittivity of free space})$$

$$\mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2 \quad (\text{permeability of free space})$$

$$c = 3.00 \times 10^8 \text{ m/s} \quad (\text{speed of light})$$

$$e = 1.60 \times 10^{-19} \text{ C} \quad (\text{charge of the electron})$$

$$m = 9.11 \times 10^{-31} \text{ kg} \quad (\text{mass of the electron})$$

## SPHERICAL AND CYLINDRICAL COORDINATES

## Spherical

$$\begin{cases} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{cases} \quad \begin{cases} \hat{\mathbf{x}} = \sin \theta \cos \phi \hat{\mathbf{r}} + \cos \theta \cos \phi \hat{\boldsymbol{\theta}} - \sin \phi \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin \theta \sin \phi \hat{\mathbf{r}} + \cos \theta \sin \phi \hat{\boldsymbol{\theta}} + \cos \phi \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}(\sqrt{x^2 + y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \quad \begin{cases} \hat{\mathbf{r}} = \sin \theta \cos \phi \hat{\mathbf{x}} + \sin \theta \sin \phi \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos \theta \cos \phi \hat{\mathbf{x}} + \cos \theta \sin \phi \hat{\mathbf{y}} - \sin \theta \hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}} \end{cases}$$

## Cylindrical

$$\begin{cases} x = s \cos \phi \\ y = s \sin \phi \\ z = z \end{cases} \quad \begin{cases} \hat{\mathbf{x}} = \cos \phi \hat{\mathbf{s}} - \sin \phi \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin \phi \hat{\mathbf{s}} + \cos \phi \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \quad \begin{cases} \hat{\mathbf{s}} = \cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

Ohm's Law:  $\mathbf{J} = \sigma \mathbf{E} = (1/\rho) \mathbf{E}$ . Continuity Equation:  $\nabla \cdot \mathbf{J} = -\partial \rho / \partial t$

Mutual Inductance:  $\text{emf}_2 = -\frac{d\Phi_2}{dt} = -M \frac{dI_1}{dt}$  Self Inductance:  $\text{emf}_1 = -\frac{d\Phi_1}{dt} = -L \frac{dI_1}{dt}$

Maxwell Equations in integral form:

$$\begin{aligned} \int_S \mathbf{D} \cdot d\mathbf{a} &= Q_{\text{enclosed}} & \int_S \mathbf{B} \cdot d\mathbf{a} &= 0 \\ \int_P \mathbf{E} \cdot d\mathbf{l} &= -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{a} & \int_P \mathbf{H} \cdot d\mathbf{l} &= I_{\text{enclosed}} + \frac{d}{dt} \int_S \mathbf{D} \cdot d\mathbf{a} \end{aligned}$$

Energy density:  $u = \frac{1}{2}(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2)$  Poynting vector:  $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Boundary conditions at the interface of two materials.

$$\begin{aligned} D_1^\perp - D_2^\perp &= \sigma_f & B_1^\perp - B_2^\perp &= 0 \\ \mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel &= 0 & \mathbf{H}_1^\parallel - \mathbf{H}_2^\parallel &= \mathbf{K}_f \times \hat{\mathbf{n}} \end{aligned}$$

Wave equation in vacuum:

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

Speed of light in vacuum  $c = 1/\sqrt{\epsilon_0 \mu_0}$ . In dielectric  $v = \frac{c}{n}$  where  $n = \sqrt{\epsilon \mu / \epsilon_0 \mu_0}$ .

Plane wave solution propagating in direction  $\hat{\mathbf{k}}$  (not in metals)

$$\mathbf{E} = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta) \quad \mathbf{B} = \frac{1}{v} \hat{\mathbf{k}} \times \mathbf{E}$$

with  $\mathbf{E}_0$  perpendicular to  $\hat{\mathbf{k}}$  and  $v = \omega/k$ .

Snell's law:  $\frac{\sin \theta_T}{\sin \theta_I} = \frac{n_1}{n_2}$ .

For linearly polarized waves with polarization in the material boundary plane (not in metals):

$$E_{0R} = \frac{\alpha - \beta}{\alpha + \beta} E_{0I} \quad E_{0T} = \frac{2}{\alpha + \beta} E_{0I} \quad \alpha = \frac{\cos \theta_T}{\cos \theta_I} \quad \beta = \frac{\mu_1 v_1}{\mu_2 v_2}$$

Wave equation in metal:

$$\nabla^2 \mathbf{E} = \mu \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu \sigma \frac{\partial \mathbf{E}}{\partial t} \quad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} + \mu \sigma \frac{\partial \mathbf{B}}{\partial t}$$

Linearly polarized solution for propagation in  $z$  direction (metals):

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0 e^{-\kappa z} e^{i(kz - \omega t)} = \tilde{E}_0 e^{-\kappa z} e^{i(kz - \omega t)} \hat{x} \quad \tilde{\mathbf{B}} = \tilde{\mathbf{B}}_0 e^{-\kappa z} e^{i(kz - \omega t)} = \frac{\tilde{k}}{\omega} \tilde{E}_0 e^{-\kappa z} e^{i(kz - \omega t)} \hat{y}$$

$$\tilde{k} = k + i\kappa \quad k = \omega \sqrt{\frac{\epsilon \mu}{2}} \left[ \sqrt{1 + \left(\frac{\sigma}{\epsilon \omega}\right)^2} + 1 \right]^{1/2} \quad \kappa = \omega \sqrt{\frac{\epsilon \mu}{2}} \left[ \sqrt{1 + \left(\frac{\sigma}{\epsilon \omega}\right)^2} - 1 \right]^{1/2}$$

Reflection from conductive surface at  $90^\circ$  incidence:

$$\tilde{\mathbf{E}}_{0R} = \frac{1 - \tilde{\beta}}{1 + \tilde{\beta}} \tilde{\mathbf{E}}_{0I} \quad \tilde{\mathbf{E}}_{0T} = \frac{2}{1 + \tilde{\beta}} \tilde{\mathbf{E}}_{0I} \quad \tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega} \tilde{k}_2$$