PHYSICS 110B MIDTERM

PROBLEM 1

(a) $\phi = AB\cos\omega t$ $\quad$ $emf = -\frac{d\phi}{dt} = \omega AB\sin\omega t$

$$emf = L\frac{dI}{dt} + IR$$

write $I(t) = I_0 \sin(\omega t + \delta)$

gives $\omega AB\sin\omega t = \omega L I_0 \cos(\omega t+\delta) + I_0 R \sin(\omega t+\delta)$

$$\omega AB\sin\omega t = \omega L I_0 \cos\omega t\cos\delta - \omega L I_0 \sin\omega t\sin\delta + I_0 R\sin\omega\cos\delta + I_0 R\cos\omega t\sin\delta$$

Set the coefficients of $\sin\omega t$ and $\cos\omega t$ to be the same on both sides

$$\begin{cases} \omega AB = I_0 R\cos\delta - \omega L I_0 \sin\delta & (1) \\ 0 = \omega L I_0\cos\delta + I_0 R\sin\delta & (2) \end{cases}$$

From (2) $\boxed{\tan\delta = -\omega L/R}$,

which means $\sin\delta = \pm\frac{\omega L}{K}$ $\quad$ $\cos\delta = \mp\frac{R}{K}$

and $K = \sqrt{\omega^2 L^2 + R^2}$

(Both signs work in principle, let's see what we get)

Equation I becomes

$$kwAB = \mp I_0 R^2 \mp I_0 w^2 L^2 = \mp K^2 I_0$$

$$\boxed{\tan\delta = -\frac{wL}{R}}$$

$$\boxed{I_0 = \mp \frac{wAB}{\sqrt{w^2L^2+R^2}}}$$

$$\boxed{I(t) = I_0 \sin wt + \delta}$$

Both signs work depending on the sign we pick for $\sin\delta$ and $\cos\delta$

This could also have been done with complex numbers

$$emf = i w A B e^{iwt}$$

(The real part is $ABw\sin wt$)

$$emf = wAB\, e^{i(wt+\pi/2)}$$

Taking $I = I_0 e^{i(wt+\delta)}$

The differential equation becomes

$$iwLI_0 e^{i(wt+\delta)} + I_0 R\, e^{i(wt+\delta)} = wAB\, e^{i(wt+\pi/2)}$$

$$LI_0\, e^{i(\delta+\pi/2)} + I_0 R\, e^{i\delta} = wAB\, e^{i\pi/2}$$

Taking real and imaginary parts

$$\begin{cases} wLI_0 \cos(\delta+90^\circ) + I_0 R\cos\delta = 0 \\ LI_0 \sin(\delta+90^\circ) + I_0 R \sin\delta = wAB \end{cases}$$

$$\begin{cases} \omega L I_0 \sin\delta + I_0 R \cos\delta = 0 \\ L I_0 \cos\delta + I_0 R \sin\delta = \omega A B \end{cases}$$

$$\tan\delta = -R/\omega L$$

(This does not look the same as before but careful. The solution we picked is $\cos(\omega t+\delta)$)

This then gives

$$\boxed{I_0 = \pm \frac{\omega A B}{\sqrt{\omega^2 L^2 + R^2}}}$$

$$\boxed{\begin{array}{c} I(t) = I_0 \cos(\omega t + \delta) \\ \tan\delta = -R/\omega L \end{array}}$$

(b) Power $= \dfrac{dW}{dt} = I^2 R$ (dissipated)

$dW =$ work done by torque $= \tau\, d\theta$

But $\omega = \dfrac{d\theta}{dt} \Rightarrow dW = \omega \tau\, dt$

$$\Rightarrow \omega\tau = I^2 R = \frac{\omega^2 A^2 B^2}{\omega^2 L^2 + R^2} \sin^2(\omega t + \delta)$$

$$\boxed{\tau = \frac{\omega A^2 B^2 R}{\omega^2 L^2 + R^2} \sin^2(\omega t + \delta)}$$

PROBLEM 2

On the first coil $\quad emf = I_1 R + L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt}$

On the second coil $\quad 0 = I_2 R + L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt}$

At $t=0$, $I_1 = I_2 = 0$

This then gives $\frac{dI_2}{dt} = -\frac{M}{L_2} \frac{dI_1}{dt}$

So $\quad emf = L_1 \frac{dI_1}{dt} + M \left(-\frac{M}{L_2} \frac{dI_1}{dt} \right)$

$$emf = \left(L_1 - M^2/L_2 \right) \frac{dI_1}{dt}$$

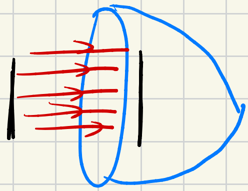
$$\boxed{L' = L_1 - M^2/L_2}$$

PROBLEM 3

We have cylindrical symmetry

Consider the blue closed surface

E-field lines in red



Here I drew the circular part of the surface with $r > a$ but could also be $r < a$

$$\oint \vec{B}\, d\vec{l} = \mu_0 \varepsilon_0 \int \frac{\partial \vec{E}}{\partial t}\, d\vec{a}$$

(a) $E = \frac{V}{d} = \frac{V}{d_0 + d_1 \sin\omega t}$ $\quad \frac{dE}{dt} = -\frac{V d_1 \cos\omega t}{(d_0 + d_1 \sin\omega t)^2}$

$$\oint \vec{B}\, d\vec{l} = 2\pi r B_\phi$$

For $r > a$ $\quad \int \frac{d\vec{E}}{dt}\, da = \frac{dE}{dt} \pi a^2$

$$B_\phi = \frac{\mu_0 \varepsilon_0}{2\pi r} \pi a^2 \left(-\frac{V d_1 \cos\omega t}{(d_0 + d_1 \sin\omega t)^2} \right)$$

$$\boxed{B_\phi = -\frac{\mu_0 \varepsilon_0 a^2}{2r} \frac{V d_1 \cos\omega t}{(d_0 + d_1 \sin\omega t)^2}}$$ for $r > a$

For $r < a$ $\quad \int \frac{d\vec{E}}{dt}\, d\vec{a} = \frac{dE}{dt} \pi r^2$

$$\boxed{B_\phi = -\mu_0 \varepsilon_0 r \frac{V d_1 \cos\omega t}{2(d_0 + d_1 \sin\omega t)}}$$ for $r < a$

(b) $Q = \text{constant}$

The electric field in $E = \frac{1}{\varepsilon_0}\frac{Q}{A}$

E is constant

$$\boxed{B = 0}$$

PROBLEM 4

(a) Need $\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t^2}$

$$\frac{\partial^2 \vec{E}}{\partial t^2} = -\omega^2 \vec{E} \qquad \frac{\partial^2 E}{\partial x^2} = k^2 \vec{E} \qquad \frac{\partial \vec{E}}{\partial y^2} = 0 \qquad \frac{\partial^2 E}{\partial z^2} = -h^2 \vec{E}$$

$$\Rightarrow k^2\vec{E} - h^2\vec{E} = -\frac{\omega^2}{c^2}\vec{E}$$

$$\boxed{h^2 - k^2 = \frac{\omega^2}{c^2}}$$

(b) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & E_y & 0 \end{vmatrix} = \frac{\partial E_y}{\partial x}\hat{z} - \frac{\partial E_y}{\partial z}\hat{x}$$

$$\vec{\nabla} \times \vec{E} = -kE_y\,\hat{z} + ihE_y\,\hat{x}$$

$$\frac{\partial B}{\partial t} = \left(kE_0\hat{z} + iE_0h\hat{x}\right)\exp\left[i(hz - \omega t) - kx\right]$$

$$\vec{B} = E_0 \frac{(k\hat{z} + ih\hat{x})}{-i\omega} \exp\left[i(hz - \omega t) - kx\right]$$

$$\boxed{B = E_0 \frac{(ik\hat{z} - h\hat{x})}{\omega} \exp\left[i(hz - \omega t) - kx\right]}$$