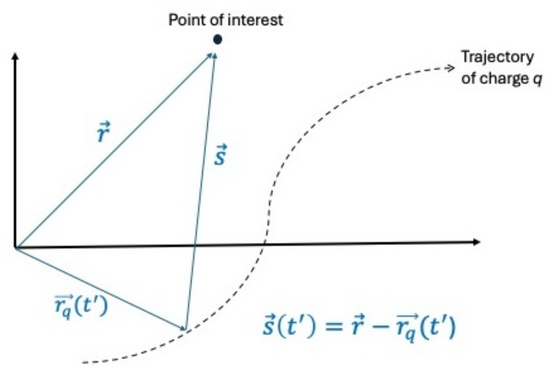
Alternative derivation Lienard-Wiechert potential

Claudio C.

The Lienard-Wiechter potential is derived in Section 10.3.1 of Griffiths. Here I present an alternative derivation from the book by Head and Marion.

We are interested in the electric potential $V(\vec{r}, t)$ generated by a moving charge q, see sketch below.



The starting point could be the Lorenz gauge potential given in equation 10.26 in Griffiths:

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t_r)}{s} d\tau' \tag{1}$$

where $\vec{s} = \vec{r} - \vec{r}\,'$ and $t_r = t - s/c$ is the retarded time. Griffiths uses the symbol $\mathscr{r}$ instead of s, but I used s in lecture to make it clearer on the blackboard and I will stick with my notation. Here I also use $\vec{r}_q$ instead of $\vec{r}\,'$ to indicate the position of the charge because we are now dealing with a single charge, not a charge distribution.

For a single charge it is more useful to recast equation 1 as

$$V(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \int \frac{\delta(t' - t_r)}{s} dt' \tag{2}$$

As shown in the sketch, s is a function of t' and the delta function picks up the values of s, *i.e.*, of the distance between the moving particle and the point of interest, at the appropriate time $t_r = t - s(t')/c$.

Applying the δ function to the integral is not trivial because t_r is itself a complicated function t'. Therefore we change variables of integration from t' to $t'' = t' - t + s/c$, so that the delta function becomes $\delta(t' - t_r) = \delta(t' - t + s/c) = \delta(t" + t - s/c - t + s/c) = \delta(t'')$.

For the change of variables, we also need the relationship between dt'' and dt'.

$$\begin{aligned} dt'' &= d(t' - t + \frac{1}{c}s) \\ dt'' &= dt' + \frac{1}{c}ds \\ dt'' &= dt'\left(1 + \frac{1}{c}\frac{ds}{dt'}\right) \end{aligned} \tag{3}$$

Now we need ds/dt'. We start from

$$s = \sqrt{\sum_i (r_i - r_{qi})^2} \tag{4}$$

where the index i runs through the three Cartesian coordinates x, y, z or $1, 2, 3$. Using the chain rule:

$$\frac{ds}{dt'} = \sum_i \frac{dr_{qi}}{dt'}\frac{\partial s}{\partial r_{qi}} = \vec{v} \cdot \vec{\nabla}_q s \tag{5}$$

where v is the velocity of the charge and $\vec{\nabla}_q s$ is the gradient of s with respect to the r_q coordinates. In a previous lecture we showed that $\vec{\nabla} s = \hat{s}$ (see also the Appendix), where in that case the gradient was taken with respect to the r_i coordinates. Given the minus sign present in equation 4, we have $\vec{\nabla}_q s = -\hat{s}$ leading to

$$\frac{ds}{dt'} = -\vec{v} \cdot \hat{s} \tag{6}$$

Substituting into equation 3 and defining $\vec{\beta} = \vec{v}/c$ we get

$$\begin{aligned} dt'' &= dt'(1 - \vec{\beta} \cdot \hat{s}) \\ dt' &= \frac{dt''}{1 - \vec{\beta} \cdot \hat{s}} \end{aligned} \tag{7}$$

We are now finally ready to do the change of variables from t' to t'' in equation 2:

$$V(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \int \frac{\delta(t'')}{s} \frac{dt''}{1 - \vec{\beta} \cdot \hat{s}} \quad = \quad \frac{1}{4\pi\epsilon_0} \frac{q}{(s - \vec{\beta} \cdot \vec{s})} \tag{8}$$

This is the Lienard-Wiechert potential given in equation 10.46 of Griffiths. The exact same procedure yields the vector potential $\vec{A}$ of equation 10.47.

A Appendix

$$s^2 = \left(\sum_i (r_i - r_{qi}) \right)^2$$

$$\vec{\nabla}(s^2) = \sum_i 2(r_i - r_{qi})\ \hat{r_i}$$

$$\vec{\nabla}(s^2) = 2\vec{s}$$

Since

$$\frac{\partial}{\partial r_i}(s^n) = n s^{n-1} \frac{\partial s}{\partial r_i}$$

we have

$$\vec{\nabla}(s^2) = 2s\vec{\nabla}s$$

$$2\vec{s} = 2s\vec{\nabla}s$$

$$\vec{\nabla}s = \hat{s}$$