EM fields from charge moving with constant velocity.

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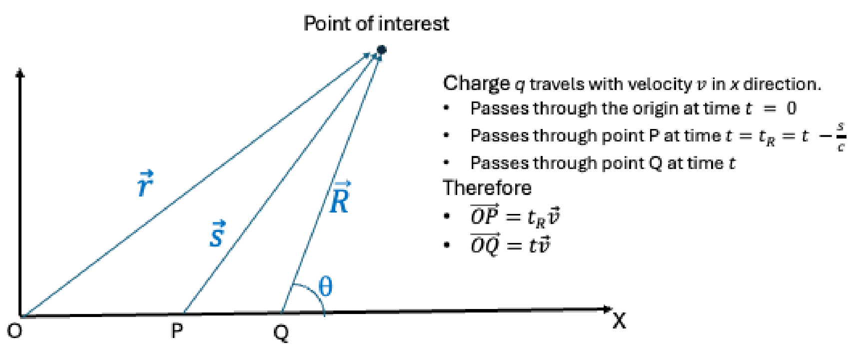
This is a more complete derivation of example 10.4 in Griffiths, *i.e.*, the EM fields generated by a particle moving with constant velocity.

The sketch below sets the stage and defines the various quantities. We are interested, to start, in the electric field at the point of interest and at time t, *i.e.*, $\vec{E}(\vec{r}, t)$.



The starting point is equation 10.72 in Griffiths, which gives the electric field as a function of velocity $\vec{v}(t_R)$ and acceleration $\vec{a}(t_R)$

$$\vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{s}{(\vec{s} \cdot \vec{u})^{3/2}} \left[(c^2 - v^2)\vec{u} + \vec{s} \times (\vec{u} \times \vec{a})\right]$$

where $\vec{u} = c\hat{s} - \vec{v}$. In our case $\vec{a} = 0$. Thus

$$\vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{(c^2 - v^2)}{(\vec{s} \cdot \vec{u})^{3/2}} \quad s\vec{u} \tag{1}$$

From the sketch we see that

$$\begin{aligned} \vec{r} &= \vec{OP} + \vec{s} = t_R\vec{v} + \vec{s} \\ \vec{s} &= \vec{r} - t_R\vec{v} \\ \vec{s} &= \vec{r} - (t - \frac{s}{c})\vec{v} \end{aligned} \tag{2}$$

Now let's look at the quantity $s\vec{u}$ which enters in equation (1). Note that the $\vec{E}(\vec{r},t)$ is in the direction of $s\vec{u}$.

$$\begin{aligned} s\vec{u} &= c\vec{s} - s\vec{v} \\ &= c\vec{r} - ct\vec{v} + s\vec{v} - s\vec{v} \\ &= c(\vec{r} - t\vec{v}) \\ &= c(\vec{r} - \vec{OQ}) \\ &= c\vec{R} \end{aligned} \tag{3}$$

This is a remarkable result. We find that $\vec{E}(\vec{r},t)$ is in the direction of $\vec{R}$, *i.e.*, the vector that joins the position of interest with the position of the charge at time t, **not at the retarded time $\boldsymbol{t_R}$**. This is probably not what you expected to find!

Next, in order to finish the job, we need to work out the quantity $(\vec{s}\cdot\vec{u})^{3/2}$ which appears in Equation 1.

Note that the perpendicular components of $\vec{s}$ and $\vec{R}$ are the same. This implies that

$$\begin{aligned} |\vec{s}\times\vec{v}|^2 &= |\vec{R}\times\vec{v}|^2 \\ s^2v^2 - (\vec{s}\cdot\vec{v})^2 &= R^2v^2 - (\vec{R}\cdot\vec{v})^2 \\ s^2v^2 - (\vec{s}\cdot\vec{v})^2 &= R^2v^2 - R^2v^2\cos^2\theta \end{aligned} \tag{4}$$

where I used the identity $|\vec{a}\times\vec{b}|^2 = a^2b^2 - (\vec{a}\cdot\vec{b})^2$.

Taking the square of equation 3:

$$\begin{aligned} c^2R^2 &= (c\vec{s} - s\vec{v})^2 \\ c^2R^2 &= c^2s^2 + s^2v^2 - 2cs\vec{s}\cdot\vec{v} \\ c^2s^2 - 2cs\vec{s}\cdot\vec{v} &= c^2R^2 - s^2v^2 \end{aligned} \tag{5}$$

Shifting our attention to $\vec{s}\cdot\vec{u}$, which is what we want:

$$\begin{aligned} \vec{s}\cdot\vec{u} &= \vec{s}\cdot(c\hat{s} - \vec{v}) = cs - \vec{s}\cdot\vec{v} \\ (\vec{s}\cdot\vec{u})^2 &= c^2s^2 - 2cs\vec{s}\cdot\vec{v} + (\vec{s}\cdot\vec{v})^2 \end{aligned}$$

Using equation 5 this becomes:

$$(\vec{s}\cdot\vec{u})^2 = c^2R^2 - s^2v^2 + (\vec{s}\cdot\vec{v})^2$$

and using equation (4) this becomes:

$$\begin{aligned} (\vec{s}\cdot\vec{u})^2 &= c^2R^2 - R^2v^2 + R^2v^2\cos^2\theta \\ &= c^2R^2\left[1 + \beta^2(\cos^2\theta - 1)\right] \\ &= c^2R^2(1 - \beta^2\sin^2\theta) \end{aligned}$$

were $\beta = v/c$. Now we can finally write:

$$(\vec{s} \cdot \vec{u})^3 = c^3 R^3 (1 - \beta^2 \sin^2 \theta)^{3/2} \tag{6}$$

Putting eqation 3 and equation 6 into equation 1:

$$\vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \; \frac{c^2 - v^2}{c^3 R^3 (1 - \beta^2 \sin^2 \theta)^{3/2}} \; c\vec{R}$$

$$\boxed{\vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \; \frac{(1 - \beta^2)}{R^2 (1 - \beta^2 \sin^2 \theta)^{3/2}} \; \hat{R}}$$

which is equation 10.75 in Griffiths. The magnetic field is given by equation 10.73 in Griffiths as

$$\vec{B}(\vec{r}, t) = \frac{1}{c} \hat{s} \times \vec{E}(\vec{r}, t)$$

which becomes

$$\vec{B}(\vec{r}, t) = \frac{1}{cs} \vec{s} \times \vec{E}(\vec{r}, t)$$

$$\vec{B}(\vec{r}, t) = \frac{1}{cs} (\vec{r} - t\vec{v} + \frac{s}{c}\vec{v}) \times \vec{E}(\vec{r}, t) \qquad \text{(using equation 2)}$$

$$\vec{B}(\vec{r}, t) = \frac{1}{cs} (\vec{R} + \frac{s}{c}\vec{v}) \times \vec{E}(\vec{r}, t) \qquad \text{(using the last three lines of equation 3)}$$

$$\boxed{\vec{B}(\vec{r}, t) = \frac{1}{c^2} \vec{v} \times \vec{E}(\vec{r}, t)}$$

which is equation 10.76 in Griffiths (in the last step I used the fact that $\vec{R} \times \vec{E}(\vec{r}, t) = 0$ since the electric field is in $\vec{R}$ direction).