# Complement to the lecture of October 10th

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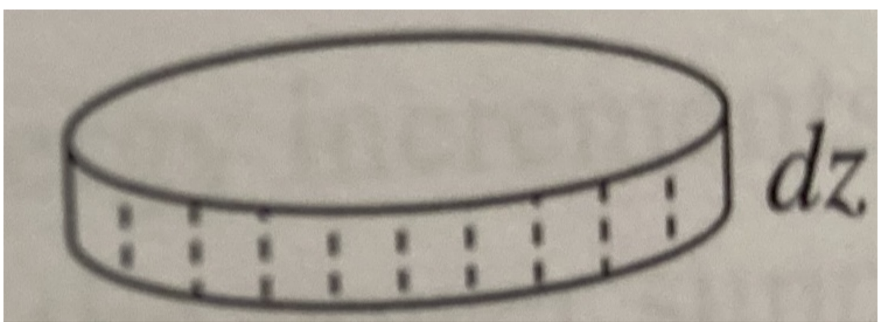
In the October 10th lecture I went through Example 8.5 on page 380 in the 5th edition of Griffiths. In the 4th edition the same example is worked out in Section 8.3 at the bottom of page 373 and continues until the middle of page 375.

In class I said that the explanation in Griffiths for the change in flux as the ring moves from $z$ to $z + dz$:

$$d\Phi = B_\rho \; 2\pi a \; dz \tag{1}$$

was not clear and that I was able to derive the same equation also in a different way. (Note: Griffiths seems to use $s, \phi, z$ for cylindrical coordinates. I have never seen this notation before. The usual notation is $\rho, \phi, z$ or even $r, \phi, z$, although this last choice can cause confusion with the $r$ variable in spherical coordinates. I guess Griffith uses $s$ instead of $\rho$ to avoid confusion with charge density. But I prefer to use common notation). Here $B_\rho$ is the $\rho$ component of the magnetic field at coordinates $\rho = a$ and $z = z$, and where we assumed cylindrical symmetry of $\vec{B}$, *i.e.*, no $\phi$ dependence, $\vec{B} = \vec{B}(\rho, z)$. This is an assumption that is not clearly stated.

First let me explain Griffiths method. Below I reproduce Figure 8.9 from the 5th edition (8.8 from the 4th edition):




This figure shows an **imaginary** cylinder with the bottom (top) face given by the initial (final) position of the ring.

For calculating the initial flux through a surface bounded by the initial position of the ring, we make the "obvious" choice of the plane bounded by the horizontal circle at the initial z position, that is to say the bottom face of this imaginary cylinder. Thus the initial flux is $\Phi_i = \Phi_B$, where $\Phi_B$ is the flux through the bottom face.

For calculating the final flux through a surface bounded by the final position of the ring, we make a less obvious choice. We choose the full surface of the cylinder except the top face. Thus the final flux is $\Phi_f = \Phi_B + \Phi_S$, where now $\Phi_S$ is the flux through the side of the cylinder (what Griffiths calls "the ribbon").

Thus, the change in flux is:

$$d\Phi = \Phi_B - (\Phi_B + \Phi_S) = \Phi_S = B_\rho \; 2\pi a \; dz \tag{2}$$

since the area of the side is the area of the circle ($2\pi a$) times the height of the cylinder ($dz$). So this explains Griffiths' claim that $d\Phi = \Phi_S$.

Here is a different way to arrive at the same result. I pick the two areas to calculate the flux in the more obvious way, *i.e.*, $\Phi_i = \Phi_B$ and $\Phi_f = \Phi_T$, where T is the flux through the top of the cylinder. So, up to a sign[1]:

$$\begin{aligned}
\Phi_B &= \int_{\phi=0}^{\phi=2\pi} \int_{\rho=0}^{\rho=a} B_z(z,\rho) \; \rho \; d\rho \; d\phi = 2\pi \int_{\rho=0}^{\rho=a} B_z(z,\rho) \; \rho \; d\rho \\
\Phi_T &= \int_{\phi=0}^{\phi=2\pi} \int_{\rho=0}^{\rho=a} B_z(z+dz,\rho) \; \rho \; d\rho \; d\phi = 2\pi \int_{\rho=0}^{\rho=a} B_z(z+dz,\rho) \; \rho \; d\rho
\end{aligned}$$

Then

$$d\Phi = \Phi_T - \Phi_B = 2\pi dz \int_{\rho=0}^{\rho=a} \frac{\partial B_z(z,\rho)}{\partial z} \; \rho \; d\rho \tag{3}$$

Now we use the fact that $\nabla \cdot \vec{B} = 0$. Since there is no $\phi$ dependence this means that

$$\begin{aligned}
\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_\rho) + \frac{\partial B_z}{\partial z} &= 0 \\
\frac{\partial B_z}{\partial z} &= -\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_\rho)
\end{aligned}$$

I can then use this equation to substitute the partial wrt $z$ in equation 3. I get

$$\begin{aligned}
d\Phi &= -2\pi dz \int_{\rho=0}^{\rho=a} \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_\rho(z,\rho)) \; \rho \; d\rho \\
d\Phi &= -2\pi dz \int_{\rho=0}^{\rho=a} \frac{\partial}{\partial \rho}(\rho B_\rho(z,\rho)) d\rho \\
d\Phi &= -2\pi dz \quad \rho B_\rho(z,\rho)\Big|_{\rho=0}^{\rho=a} \\
d\Phi &= -2\pi a \; dz \; B_\rho(a,z)
\end{aligned}$$

which, up to a sign, is the same as equation 1.

[1] Up to a sign because it depends on what we define as the positive vs negative flux, and I am not going to worry about that right now. FWIW Griffiths is cavalier about the sign as well. He never tells us whether it is the flux into the ribbon or out of the ribbon, so I am in good company here.