Polarizations for reflected and transmitted waves

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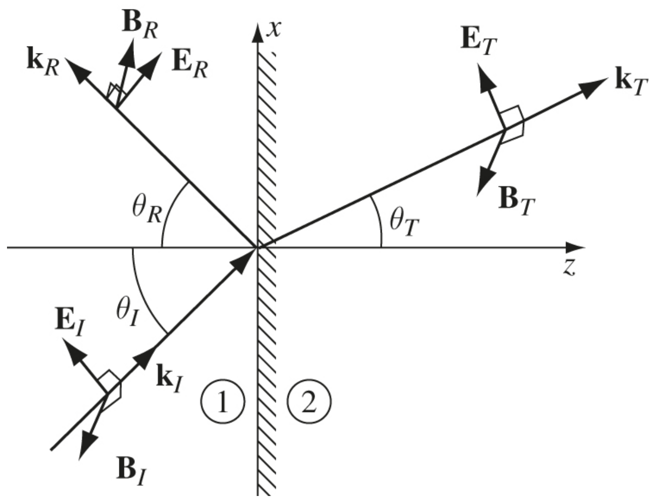
1 Introduction

Chapter 9.3.3 of Griffiths discusses reflection and transmission of EM waves polarized parallel to the plane of incidence at the boundary between two dielectrics. We went through this item in class as well. Griffiths assumes that the polarization of the reflected and transmitted waves are also in the plane of incidence, see Figure 9.15 from the book below. The plane of incidence is the (xz) plane and all the electric fields are in this plane also, *i.e.*, they have no y components.



This assumption seems obvious, but it is highly non-trivial. In fact, it turns out that the polarizations of the incident (I), reflected (R), and transmitted (T) waves are the same only in two limiting cases:

- When the incident polarization is in the plane of incidence (p-polarization, where p is for parallel).

- When the incident polarization is perpendicular to the plane of incidence (s-polarization, where s is for senkrecht, which means perpendicular in German).

Griffiths, as well as a few other books that I consulted, do not justify this assumption at all. In his collected lectures, Feynman[1], gives a physics justification but also points out that (of course) this result is embedded in the boundary conditions. However, he says *"When you have some spare time, see if you can get the same result from the equations"*.

The physical argument is that oscillating electric fields set electric dipoles in motion. For an isotropic material, the dipoles oscillate in the direction of $\vec{E}$ and radiate. It turns out that the radiation from an oscillating dipole is polarized in the direction of oscillation, *i.e.*, in the direction of $\vec{E}$. The key point is that the boundary conditions for $\vec{E}$ fields parallel and perpendicular to the surface are different. Thus if the incident wave has both parallel and perpendicular components, the total $\vec{E}$ on the surface will not be in the same direction as the incoming $\vec{E}$, and thus the polarization is not preserved by the reflected and transmitted waves.

It would be nice to prove Griffiths assumption directly from the boundary conditions, as Feynman suggests. I did not find such a proof anywhere (let me know if you do), so here is my own.

2 Preliminaries

In everything that follows we will use the axes as defined in Figure 9.15 of Griffiths, and subscripts I, R, and T for the incident, reflected, and transmitted waves. All electric/magnetic fields in what follows are at the boundary between dielectrics.

Write unit vectors in the direction of propagation of the waves. We will need these later.

$$\begin{aligned} \hat{k}_I &= \sin\theta_I \hat{x} + \cos\theta_I \hat{z} \\ \hat{k}_R &= \sin\theta_R \hat{x} - \cos\theta_R \hat{z} \\ \hat{k}_T &= \sin\theta_T \hat{x} + \cos\theta_T \hat{z} \end{aligned} \tag{1}$$

We are going to work with the boundary conditions expressed in vectorial form[2].

[1] https://www.feynmanlectures.caltech.edu/II_33.html, Section 33-4

[2] This is equation 7.37 in Jackson, Classical Electrodynamics, 2nd Edition.

It may be a bit of an overkill, especially for the first two.

$$
\begin{aligned}
(i) \quad & \left[\epsilon_1 \vec{E}_I + \epsilon_1 \vec{E}_R - \epsilon_2 \vec{E}_T\right] \cdot \hat{z} = 0 \\
(ii) \quad & \left[\vec{E}_I + \vec{E}_R - \vec{E}_T\right] \times \hat{z} = 0 \\
(iii) \quad & \left[\frac{1}{v_1}(\vec{E}_I \times \hat{k}_I) + \frac{1}{v_1}(\vec{E}_R \times \hat{k}_R) - \frac{1}{v_2}(\vec{E}_T \times \hat{k}_T)\right] \cdot \hat{z} = 0 \\
(iv) \quad & \left[\frac{1}{\mu_1 v_1}(\vec{E}_I \times \hat{k}_I) + \frac{1}{\mu_1 v_1}(\vec{E}_R \times \hat{k}_R) - \frac{1}{\mu_2 v_2}(\vec{E}_T \times \hat{k}_T)\right] \times \hat{z} = 0
\end{aligned}
$$

3 Incident p-polarization

In this case the incident $\vec{E}_I$ is in the (xz) plane, *i.e.*, its y-component $E_{Iy} = 0$. We want to show that then the y-components of the reflected and transmitted electric fields must also be $E_{Ry} = E_{Ty} = 0$.

We are interested in the y-components of the electric fields. Equation (ii) gives

$$\boxed{E_{Ry} = E_{Ty}}$$

The next step is trickier. We will consider the y component of equation (iv), by taking its dot product with $\hat{y}$. This involves a bunch of terms of the form (I suppressed the I, R, and T labels):

$$\hat{y} \cdot \left[\hat{z} \times (\vec{E} \times \hat{k})\right] = \hat{y} \cdot \left[(\hat{k} \cdot \hat{z})\vec{E} - (\vec{E} \cdot \hat{z})\hat{k}\right] = \hat{y} \cdot \left[\pm \cos\theta \vec{E} - E_z \hat{k}\right] = \pm \cos\theta E_y \tag{2}$$

Where the upper (+) sign is for the I and T waves, and the lower ($-$) sign is for the R wave (see for yourself what $\hat{k} \cdot \hat{z}$ is using equation 1). In deriving this result, I also used the vector identity

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

and also the fact that $\hat{k}$ is perpendicular to $\hat{y}$.

Now we are ready to take the dot product of $\hat{y}$ with equation (iv), using the result in equation 2, and also using the facts that $E_{Iy} = 0$, $\theta_R = \theta_I$, and

$v_1/v_2 = \sin\theta_I / \sin\theta_T$:

$$\begin{aligned}
&\frac{\cos\theta_I}{\mu_1 v_1} E_{Ry} + \frac{\cos\theta_T}{\mu_2 v_2} E_{Ty} = 0 \\
&E_{Ry} = -\frac{\mu_1}{\mu_2} \frac{v_1}{v_2} \frac{\cos\theta_T}{\cos\theta_I} E_{Ty} \\
&E_{Ry} = -\frac{\mu_1}{\mu_2} \frac{\sin\theta_I}{\sin\theta_T} \frac{\cos\theta_T}{\cos\theta_I} E_{Ty} \\
&\boxed{E_{Ry} = -\frac{\mu_1}{\mu_2} \frac{\tan\theta_I}{\tan\theta_T} E_{Ty}}
\end{aligned}$$

Because of the relative minus sign, the two boxed equations are incompatible, regardless of what μ_1, μ_2, θ_I, and θ_T could be, **unless $\boldsymbol{E_{Ry} = E_{Ty} = 0}$**. QED.

4 Incident s-polarization

In this case $E_{Ix} = E_{Iz} = 0$, *i.e.*, $\vec{E}_I = E_0\hat{y}$. We want to show that then $E_{Rx} = E_{Rz} = E_{Tx} = E_{Tz} = 0$.

From (i) and (ii) we have

$$\begin{aligned}
&(a) \quad \epsilon_1 E_{Rz} = \epsilon_2 E_{Tz} \\
&(b) \quad E_{Rx} = E_{Tx}
\end{aligned} \tag{3}$$

Since the waves are transverse:

$$\begin{aligned}
&\vec{E}_R \cdot \hat{k}_R = 0 \\
&\sin\theta_R E_{Rx} - \cos\theta_R E_{Rz} = 0 \\
&E_{Rz} = \tan\theta_R E_{Rx}
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
&\vec{E}_T \cdot \hat{k}_T = 0 \\
&\sin\theta_T E_{Tx} + \cos\theta_T E_{Tz} = 0 \\
&E_{Tz} = -\tan\theta_T E_{Tx} \\
&E_{Rz} = -\frac{\epsilon_2}{\epsilon_1} \tan\theta_T E_{Rx}
\end{aligned} \tag{5}$$

where in the last step I used equation 3 to change from the transmitted (T) to the reflected (R) fields. Because of the minus sign on the right hand side of equation 5, equations 4 and 5 are incompatible, regardless of what ϵ_1, ϵ_2, θ_R, and θ_T could be, **unless $\boldsymbol{E_{Rx} = E_{Rz} = 0}$**. Then, using equation 3, we also get $\boldsymbol{E_{Tx} = E_{Tz} = 0}$. QED.

5 Incident mixed polarization

In this case the reflected and transmitted wave will not have the same polarization with respect to the plane of incidence as the incoming wave. Fundamentally, this is because the p- and s-components of the incoming wave have different reflection/transmission coefficients.

We can write the incident electric field as a superposition of a p- and s-polarization:

$$\vec{E}_I = E_p \hat{p}_I + E_s \hat{y}$$

where $\hat{p}_I = \cos\theta_I \hat{x} - \sin\theta_I \hat{z} = \hat{y} \times \hat{k}_I$ (see the Figure on page 1).

The reflected field is

$$\vec{E}_R = \frac{\alpha - \beta}{\alpha + \beta} E_p \hat{p}_R + \frac{1 - \alpha\beta}{1 + \alpha\beta} E_s \hat{y}$$

where (following Griffiths Chapter 9.3.3):

- $\hat{p}_R = \cos\theta_I \hat{x} + \sin\theta_I \hat{z} = -\hat{y} \times \hat{k}_R$ (see Figure on page 1)
- $\alpha = \cos\theta_T / \cos\theta_I$ (Griffiths Eq. 9.108)
- $\beta = \frac{\mu_1 v_1}{\mu_2 v_2}$ (Griffiths Eq. 9.106)
- $\frac{\alpha-\beta}{\alpha+\beta}$ is the ratio of the reflected to the incident electric field amplitude for p-polarization. (Griffiths Eq. 9.109)
- $\frac{1-\alpha\beta}{1+\alpha\beta}$ is the same ratio but for s-polarization. (Griffiths Problem 9.17, also in Homework 4).

Comparing the expressions for $\vec{E}_I$ and $\vec{E}_R$ we can say that the polarization with respect to the plane of incidence is preserved only if $\frac{\alpha-\beta}{\alpha+\beta} = \frac{1-\alpha\beta}{1+\alpha\beta}$, *i.e.*, $\alpha = 1$, *i.e.*, $\theta_T = \theta_I$, which happens only at normal incidence $\theta_I = 0$. This is not surprising because in this case the p- and s-polarizations are indistinguishable[3]. In fact, at normal incidence the plane of incidence cannot even be defined. A similar argument can be applied to the transmitted wave. Therefore we conclude that **in the oblique incidence case polarization is only preserved for pure p- or s-polarization.**

[3]In our convention p-polarization at normal incidence corresponds to $\vec{E}_I = E\ \hat{x}$, and s-polarization corresponds to $\vec{E}_I = E\ \hat{y}$. In both cases, at normal incidence, $\vec{E}_I$ is parallel to the surface. These two cases are then physically the same.