15.43

(a) \( dx = (d\vec{x}, dt) \) is a 4-vector

\[ dx = (\vec{v}, c)dt \]

\[ dx^2 = (\vec{v}^2 - c^2)dt^2 \]

If \( v < c \) in a given frame \( \Rightarrow dx^2 < 0 \)

But \( dx^2 \) is invariant \( \Rightarrow dx^2 < 0 \) in all frames \( \Rightarrow v < c \) in all frames

(b) Same argument as before, but replace the "<" sign with the "=" sign

15.49

\( u = (\sqrt{\frac{\eta^2}{2}}, \sqrt{\frac{2}{\eta^2}}) \)

\( u \cdot u = \eta^2 \frac{\eta^2}{2} - \eta^2 \frac{2}{\eta^2} = \eta^2 \frac{\eta^2}{2} (1 - \frac{\eta^2}{2}) = -\eta^2 \)

15.52

Initial 4-momentum = \( P_{\text{INIT}} \)

Final 4-momentum = \( P_{\text{FIN}} \)

Change in 4-momentum \( \vec{Q} = P_{\text{FIN}} - P_{\text{INIT}} = (\vec{0}, Q_4) \)

\( \vec{Q} = (Q_1, Q_2, Q_3) \) is zero in all frames

Boost \( \vec{Q} \) into a different frame \( (V_0 \text{ along } x\text{-axis}) \)

\[
\begin{align*}
Q'_1 &= \gamma (Q_1 - \frac{\vec{v} \cdot \vec{Q}}{c^2} Q_4) \\
Q'_4 &= \gamma (Q_4 - \frac{\vec{v}}{c} Q_1)
\end{align*}
\]
But \( Q_1 = 0 \) and \( Q'_1 = 0 \). Therefore

\[
\begin{align*}
0 &= -\gamma \frac{e}{c} Q_4 \\
Q'_4 &= \gamma Q_4
\end{align*}
\]

\( \Rightarrow \) \( Q_4 = 0 \) \( \Rightarrow \) \( Q'_4 = 0 \)

\( \Rightarrow \) 4th component of \( p \) conserved

**15.53**

In rest frame of \( a \), \( p_A = (0, m_A c) \)

\( p_B = (\vec{p}_B, \frac{E_B}{c}) \)

\( \Rightarrow p_A \cdot p_B = -m_A E_B \checkmark \)

In the same way, in the rest frame of \( b \), \( p_A \cdot p_B = -m_B E_A \checkmark \)

In the rest frame of \( a \), \( E_B = \gamma m_B c^2 \) with \( \gamma = \frac{1}{\sqrt{1 - (\frac{v_{\text{rel}}}{c})^2}} \)

\( \Rightarrow p_A \cdot p_B = -\gamma (v_{\text{rel}}) m_A m_B c^2 \checkmark \)

**15.60**

Before \( \quad \) After

\[
\vspace{1cm}
\text{Before: 4-vector} \quad (\vec{0}, m_A c) = p_{in} \\
\text{After: Two 4-vectors} \quad (\vec{p}_1, \frac{E_1}{c}) \quad \text{&} \quad (\vec{p}_2, \frac{E_2}{c})
\]

The "After" 4-vector is the sum of the two

\[
p_{\text{after}} = (\vec{p}_1 + \vec{p}_2, \frac{E_1 + E_2}{c})
\]

\( p_{in} = p_{\text{after}} \) implies \( \vec{p}_1 + \vec{p}_2 = \vec{0} \) \( \Rightarrow \) \( \vec{p}_1 = -\vec{p}_2 \) \( \Rightarrow \) \( |\vec{p}_1| = |\vec{p}_2| = q \)

\[
\left( \frac{E_1 + E_2}{c} = m_A \right)
\]
\[ E_1 = \sqrt{m_A^2 + q^2} \quad \text{and} \quad E_2 = \sqrt{m_B^2 + q^2} \quad \Rightarrow \quad E_1 = E_2 = E \]

But we had \( \frac{E_1 + E}{c} = \frac{m_A}{c} \)

\[ E = \frac{1}{2} m_A c^2 \quad \text{(both B particles have this energy)} \]

But \( E = \gamma m_B c^2 \) with \( \gamma = \frac{1}{\sqrt{1 - \frac{v_B^2}{c^2}}} \) \( \Rightarrow \) \( v_B = \text{velocity of } B \text{ particle} \)

\[ 2 \gamma m_B = \frac{1}{2} m_A c^2 \]

\[ \gamma = \frac{m_A}{2 m_B} \quad 1 - \frac{v_B^2}{c^2} = \frac{2 m_B}{m_A} \quad \Rightarrow \quad v_B^2 = 1 - \frac{2 m_B}{m_A} \]

\[ \boxed{\gamma^2 = 1 - \frac{2 m_B}{m_A}} \]

15.86

(a) \( E = \frac{m_\pi c^2}{2} = 67.5 \text{ MeV} \)

(b) \( \text{cm} \)

\( E_0, E_2, E_1 \)

\( E_0 = \frac{m_\pi c^2}{2} \quad E_1 = 3E_2 \)

\( \begin{aligned} E_2 &= \gamma (E_0 - \beta E_0) \\
E_1 &= \gamma (E_0 + \beta E_0) \end{aligned} \)

Divide these two equations \( 3 = \frac{1 + \beta}{1 - \beta} \)

\( 3 - 3\beta = 1 + \beta \quad 2 = 4\beta \quad \boxed{\beta = \frac{1}{2}} \)