Poisson Limit, Frequentist vs Bayesian

Let $N_0 =$ number of observed events $\mu_{90}$ the 90% limit. The Poisson distribution of mean $\mu$ is

$$p(N|\mu) = \frac{\mu^N e^{-\mu}}{N!}$$

The frequentist limit is given by solving for $\mu_{90}$ such that

$$p(N \leq N_0|\mu_{90}) = 0.1$$

$$\sum_{i=0}^{i=N_0} p(i|\mu_{90}) = 0.1$$

$$e^{-\mu_{90}} \sum_{i=0}^{i=N_0} \frac{\mu^i}{i!} = 0.1$$

(1)

The posterior distribution for a Poisson with uniform prior is

$$p(\mu|N)d\mu = \frac{\mu^N e^{-\mu}}{N!}d\mu$$

Note that this is properly normalized

$$\int_0^{\infty} p(\mu|N) d\mu = \frac{1}{N!} \int_0^{\infty} \mu^N e^{-\mu} d\mu = \frac{1}{N!} \Gamma(N + 1) = \frac{\Gamma(N + 1)}{N!} = 1$$

where $\Gamma(a, b)$ is the upper incomplete gamma function. The 90% limit is then given by solving

$$\int_{\mu_{90}}^{\infty} p(\mu|N_0)d\mu = 0.1$$

$$\frac{1}{N_0!} \int_{\mu_{90}}^{\infty} \mu^{N_0} e^{-\mu} d\mu = 0.1$$

$$\frac{1}{N_0!} \Gamma(N_0 + 1, \mu_{90}) = 0.1$$

$$\frac{1}{N_0!} N_0! e^{-\mu_{90}} \sum_{i=0}^{i=N_0} \frac{\mu^i}{i!} = 0.1$$

(2)

where I have used the expansion of $\Gamma(a, b)$ when $a$ is a positive integer. The factorials cancel, and equations (1) and (2) are identical.