Group Theory

Imagine taking the direct product of two three-vectors. This gives a 3x3 matrix.

Imagine now that I rotate my coordinate system. How does my matrix change? To answer this, it helps to break down the matrix into several small matrices: one that is symmetric and traceless (5 degrees of freedom), one that is anti-symmetric (3 degrees of freedom), and one that is a number times the identity matrix (1 degree of freedom). Then each component transforms separately: the anti-symmetric component transforms just like a vector would; the symmetric component transforms in its own way, and the identity matrix doesn't transform at all.

We can consider this as the SO(3) group. (S just meaning that the matrix has determinant 1 rather than -1)

Similarly, consider the possibility of adding angular momenta: a particle of spin-1 has a certain probability of having its z-component be 1, 0, or -1. Again, rotating the vector will change this value, so again, this problem can be considered just rotating the vector under the O(3) group.

Now the SO(3) group is isomorphic to the SU(2) group. We're familiar with this group – we've used the Pauli matrices before. We know the Pauli Matrices have the property $[\sigma_1, \sigma_2] = 2i\sigma_3$. We say that the structure constant of SU(2) is 2i.

Are there any other sets of matrices – in any dimension – that obey this relation? If so, then they can be another representation of the SU(2) group. So yes, the SU(2) group can be represented by higher-order matrices. In fact, SU(2) is special because it can be represented by matrices of any dimensionality; not all groups have this.

We say that the Pauli Matrices are the fundamental representation of SU(2). This is just an arbitrary choice, but we choose it because it's the simplest case. Another important case is the Adjoint Representation, which has the special property that the Ath matrix's ij component is equal to the structure constant, f_{aij}.