Srednicki Chapter 9

QFT Problems & Solutions

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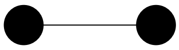
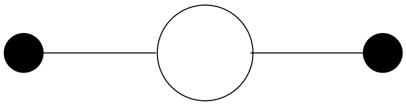
Srednicki 9.1. State and justify the symmetry factors in figure 9.13



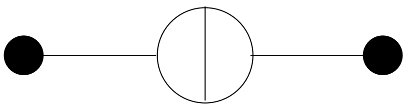
Swapping the sources is the same thing as swapping the ends of the propogator. So, we've overcounted by two. S = 2.

You may be concerned because we didn't count the sources when we did our counting. So in what sense is this an overcounting? Remember that the source, whatever it is, will just be its own collection of propogators and vertices. Whatever they are, these will be able to be exchanged with each other, and this will be equivalent to swapping the ends of the (drawn) propogator. So that's the sense in which they are overcounted.

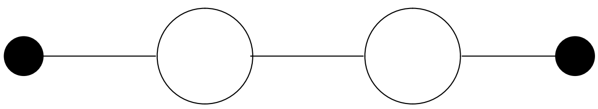
This may cause yet more consternation – if these bundles of propogators and vertices are added, won't they affect the symmetry factor? Yes – but that's OK. When the sources are specified, it will indeed become necessary to multiply our S with the S from each source.



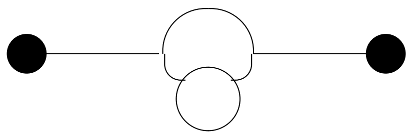
Swapping the vertices is the same as swapping the sources and reversing the two straight propogators. That's 2. Also, swapping the two curved propogators is the same as swapping two legs in each propogator. That's 2 more. S = 4.



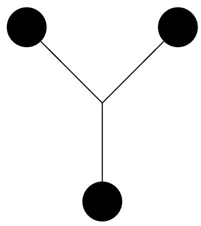
Swapping the sources is redundant as before, since swapping every other vertex and propogator except that in the middle is the same thing. That's 2. Also, swapping the two vertices at the top and bottom of the loop is the same as swapping across all the loop propogators across the horizontal diagonally, and reversing the direction of the vertical propogator. That's 2 more. S = 4.



Swapping the sources is redundant as before, since swapping a bunch of vertices and propogators is the same thing. That's 2. Also, swapping the two propogators in a loop is the same as swaping the legs of those vertices. That's 4 more, two for each loop. S = 8.

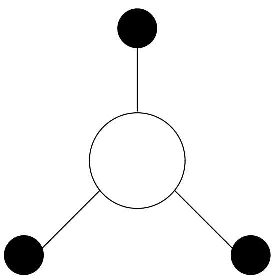


Swapping the sources is redundant as before, since swapping a bunch of vertices and propogators is the same thing. That's 2. Also, swapping the two propogators in the small loop is the same as swaping the legs of those vertices. That's 2 more. S = 4.

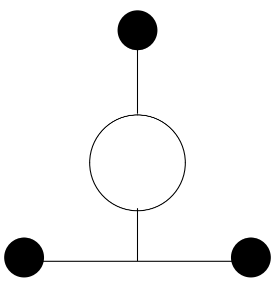


Swapping the sources (in 3! different combinations) is the same as swapping the propogators (in 3! different combinations). S = 6.

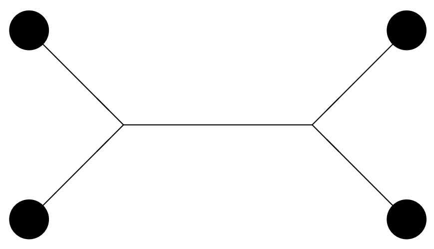
Why can't we swap the legs of the vertex as before, and get another factor of 3!? Once we eliminate the option to move the sources, the sources are no longer interchangeable. So having a leg A matched to source A and propogator B is not the same thing as having leg B matched to source A and propogator A.



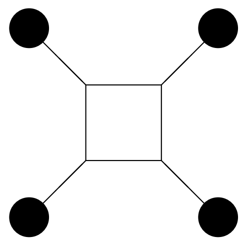
Swapping the sources (in 3! different combinations) is the same as swapping the straight propogators (in 3! different combinations). S = 6.



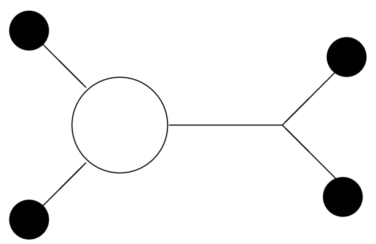
Swapping the curved propogators is the same is swapping the legs of those vertices. That's 2. Additionally, swapping the straight horizontal propogators and sources is the same as swapping the legs of the vertex. That's 2 more. S = 4.



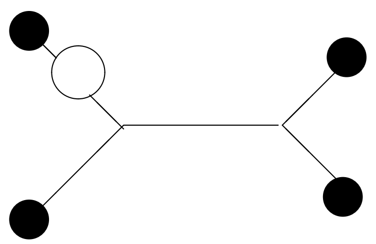
Reversing the straight propogator in the middle in the same as reversing the vertices. That's 2. Switching the two progagators on the left (or on the right) is the same as switching the legs of the vertices. That's 4. S = 8.



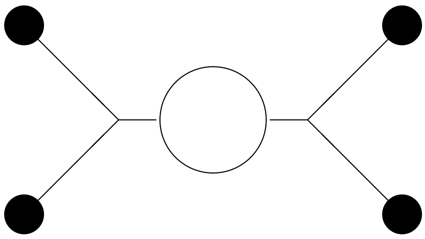
Reversing the two left propogators and sources is the same as switching the vertices. That's 2. Same on the right – that's 2 more. Finally, switching the two on the left with the two on the right is the same as switching those two vertices. That's 2 more. S = 8.



Reversing the two left propogators and sources is the same as switching the vertices. That's 2. Same on the right – that's 2 more. S = 4.



Swapping the propagators in the loop is the same as switching the legs of the vertices. That's 2. Swapping the vertices on the right (with their sources) is the same as switching the legs of the vertices. That's 2 more. S = 4.



Swapping the propagators in the loop is the same as switching the legs of the vertices. That's 2. Swapping the vertices on the right (or the left) – with their sources – is the same as switching the legs of the vertices. That's 4 more. Finally, the entire diagram to the right of the loop can be swapped with the entire diagram to the left of the loop – this is the same as switching the vertices and reversing the propogators. That's 2 more. S = 16.

Srednicki 9.2. Consider a real scalar field with the Lagrangian specified by:

$$\mathcal{L}_0 = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}\mathbf{m}^2\phi^2$$

$$\mathcal{L}_1 = -\frac{1}{24}\mathbf{Z}_\lambda\lambda\phi^4$$

$$\mathcal{L}_{\mathbf{ct}} = -\frac{1}{2}(\mathbf{Z}_\phi - 1)\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}(\mathbf{Z}_\mathbf{m} - 1)\mathbf{m}^2\phi^2$$

(a) What kind of vertex appears in the diagrams for this theory (that is, how many line segments does it join?) and what is the associated vertex factor?

The $\mathcal{L}_1$ terms specifies the interaction. The ϕ^4 term indicates that the vertex joins $\boxed{\text{four}}$ line segments. The associated vertex factor is simply $\boxed{-i\lambda\int d^4x}$. Why? The vertex factor is everything in front of the vertex. We add a factor of 4! since the vertex legs can be arranged in 4! ways. Finally, an i and $\int d^4x$ are added as in the text – this comes from the form of equation 9.5. We'll also neglect the Z_g since $Z_g = 1 + O(g^2) + \ldots$.

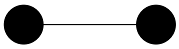
Srednicki's solution neglects the integral. This is very poorly explained in the text, but I think the best solution is to include the integral for chapter 9, and neglect it starting in chapter 10. See my chapter 10 slides for an explanation of why we neglect this integral.

(b) Ignoring the counterterms, draw all the connected diagrams with $1 \le \mathbf{E} \le 4$ and $0 \le \mathbf{V} \le 2$, and find their symmetry factors.

Consider equation 9.11. This second term is Z_0, which will not change. In the first term, though, there will be four functional derivatives. Hence, the number of surviving sources will be $E = 2P - 4V$.

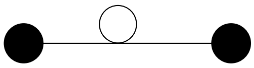
For these diagrams then, anything with E odd won't work, since that requires us to have half a propogator. We also won't have anything with E = 4, V = 0, since that corresponds to two propogators, and there's no way to combine two propogators without a vertex, so we can't have any connected diagrams. As for the others, we have:

$\mathbf{E = 2, V = 0} \implies \mathrm{P} = 1.$



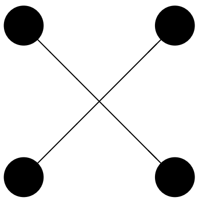
S = 2

$\mathbf{E = 2, V = 1} \implies \mathrm{P} = 3.$



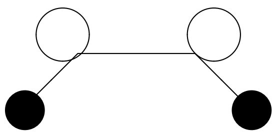
S = 2^2

$\mathbf{E = 4, V = 1} \implies \mathrm{P} = 4.$

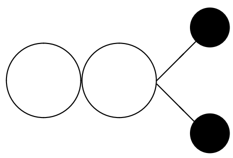


S = 4!

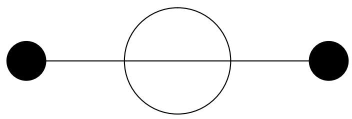
$\mathbf{E = 2, V = 2} \implies \mathrm{P} = 5.$



S = 2^3

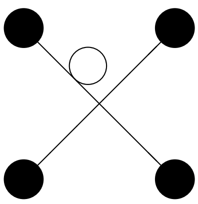


S = 2^3

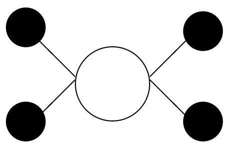


S = $2 \times 3!$

$\mathbf{E = 4, V = 2} \implies \mathrm{P} = 6.$



S = 4!



S = 2^4

(c) Explain why we did not have to include a counterterm linear in ϕ to cancel tadpoles.

The counterterm linear in ϕ is needed to cancel those terms that have a single source, with the source removed. In this case there are no terms with a single source, so this goes to zero of its own accord.

Srednicki 9.3. Consider a complex scalar field (see problems 3.5, 5.1, and 8.7)

with the Lagrangian specified by:

$$\mathcal{L}_0 = -\partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi$$

$$\mathcal{L}_1 = -\frac{1}{4} Z_\lambda \lambda (\phi^\dagger \phi)^2$$

$$\mathcal{L}_{ct} = -(Z_\phi - 1)\partial^\mu \phi^\dagger \partial_\mu \phi - (Z_m - 1) m^2 \phi^\dagger \phi$$

This theory has two kinds of sources, J and J†, and so we need a way to tell which is which when we draw the diagrmas. Rather than labeling the source blobs with a J or J†, we will indicate which is which by putting an arrow on the attached propagator that points towards the source if it is a J†, and away from the source if it is a J.

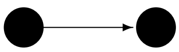
(a) What kind of vertex appears in the diagrams for this theory, and what is the associated vertex factor? Hint: your answer should involve these arrows.

The vertex joins two lines with incoming arrows, and two with outgoing arrows. The vertex factor is $2!2! \times i \int d^4x \times (-\frac{1}{4}) Z_\lambda \lambda = -i Z_\lambda \lambda \int d^4x = -i\lambda \int d^4x + \ldots$.

(b) Ignoring the counterterms, draw all the connected diagrams with 1$\leq$ E $\leq$ 4 and **0 $\leq$ V $\leq$ 2, and find their symmetry factors. Hint: the arrows are important!**

The diagrams are the same as last time, all we have to do is add the arrows and recalculate the symmetry factors.

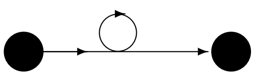
E = 2, V = 0 $\Longrightarrow$ P = 1.



S = 1

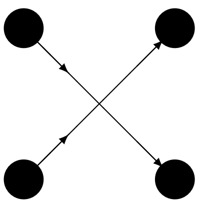
Note that S = 1 because the sources cannot be reversed in this theory: one source (with the arrow pointing away from it) is a J and the other one is a $J^\dagger$. They are not interchangeable with each other.

E = 2, V = 1 $\Longrightarrow$ P = 3.



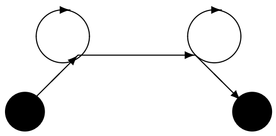
S = 1

E = 4, V = 1 $\Longrightarrow$ P = 4.

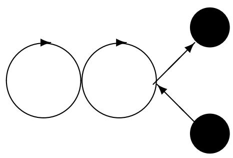


S $= 2^2$

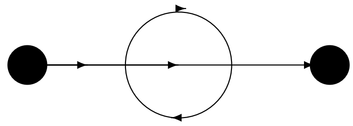
$\mathbf{E = 2, V = 2} \implies$ P $= 5$.



S $= 1$

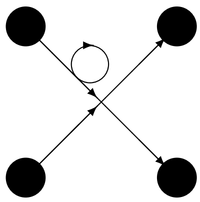


S $= 1$

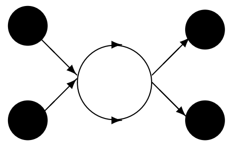


S $= 2!$

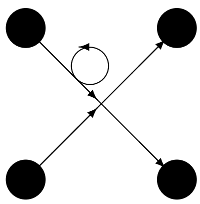
$\mathbf{E = 4, V = 2} \implies$ P $= 6$.



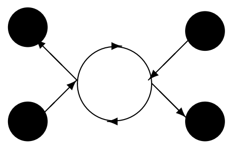
S $= 2$



S $= 2^3$



S $= 2$



S $= 2$

Srednicki 9.4. Consider the integral

$$\mathbf{exp\ W(g, J)} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}\pi}} \int_{-\infty}^{\infty} \mathbf{dx\ exp} \left[-\frac{\mathbf{1}}{\mathbf{2}}\mathbf{x^2} + \frac{\mathbf{1}}{\mathbf{6}}\mathbf{gx^3} + \mathbf{Jx} \right]$$

This integral does not converge, but it can be used to generate a joint power series in g and J,

$$\mathbf{W(g, J)} = \sum_{\mathbf{V=0}}^{\infty} \sum_{\mathbf{E=0}}^{\infty} \mathbf{C_{V,E} g^V J^E}$$

(a) Show that

$$\mathbf{C_{V,E}} = \sum_{\mathbf{I}} \frac{\mathbf{1}}{\mathbf{S_I}}$$

where the sum is over all connected Feynman diagrams with E sources and V three-point vertices, and S_I is the symmetry factor for each diagram.

We have:

$$exp\ W(g, J) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\ exp \left[-\frac{1}{2}x^2 + \frac{1}{6}gx^3 + Jx \right]$$

Breaking up this exponential:

$$exp\ W(g,J) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\ e^{-\frac{1}{2}x^2} e^{\frac{1}{6}gx^3} e^{Jx}$$

Now we will expand these last two exponentials into series:

$$exp\ W(g,J) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\ e^{-\frac{1}{2}x^2} \sum_{V=0}^{\infty} \frac{1}{V!} \left[\frac{1}{6}gx^3\right]^V \sum_{E=0}^{\infty} \frac{1}{E!} [Jx]^E$$

which is:

$$exp\ W(g,J) = \frac{1}{\sqrt{2\pi}} \sum_{V=0}^{\infty} \sum_{E=0}^{\infty} \frac{J^E g^V}{6^V V! E!} \int_{-\infty}^{\infty} dx\ e^{-\frac{1}{2}x^2} x^{3V+E}$$

This last function is odd (and integrates to zero) if 3V+E is odd. So, we will restrict the sum to terms with 3V+E is even.

$$exp\ W(g,J) = \frac{1}{\sqrt{2\pi}} \sum_{V=0}^{\infty} \sum_{\substack{E=0 \\ E+3V \text{ even}}}^{\infty} \frac{J^E g^V}{6^V V! E!} \int_{-\infty}^{\infty} dx\ e^{-\frac{1}{2}x^2} x^{3V+E}$$

This integral appears on tables of integrals (actually, the integral shown usually ranges from 0 to ∞, but this is an even function, so multiply by two). The result is:

$$exp\ W(g,J) = \sum_{V=0}^{\infty} \sum_{\substack{E=0 \\ E+3V \text{ even}}}^{\infty} \frac{J^E g^V}{6^V V! E!} (3V+E-1)!! \tag{9.4.1}$$

Now we'll switch to using diagrams. We'll sum over all possible connected diagrams, and give each diagram a factor of g for each vertex and of J for each external source. To account for some diagrams that are included more than once in the sum, we'll need to add in a multiplication factor, which will be, as in the text, $6^V V! 2^P P!$ divided by the symmetry factor S_I. Of course, there could still be cross-diagram contributions, which we'll add in momentarily.

There's one key point though. Under this prescription, we're treating every possible combination of vertex and propagator differently. In the text this was appropriate, because the surviving source would depend on the combination, and would yield different diagrams (though only the dummy indices would change). But in our case, which vertex gets paired with which propagator is irrelevent, because we'll still have the same diagram at the end (even the dummy indices will be the same). Hence, the prescription above will overcount by the number of vertex-propagator combinations – a factor of $\frac{(2P)!}{(2P-3V)!}$. We need to divide this out. Then,

$$exp\ W(g,J) = \sum_{\{I\}} \frac{1}{6^V V! E!} (3V+E-1)!! \frac{6^V V! 2^P P! (2P-3V)!}{(2P)! S_I}$$

Cancelling the obvious terms:

$$exp\ W(g,J) = \sum_{\{I\}} \frac{1}{E!} (3V+E-1)!! \frac{2^P P! (2P-3V)!}{(2P)! S_I}$$

Note that $3V + E = 2P$, so:

$$exp\ W(g,J) = \sum_{\{I\}} \frac{1}{E!}(2P-1)!!\frac{2^P P!(2P-3V)!}{(2P)!S_I}$$

Also, $2^P P! = (2P)!!$ Then,

$$exp\ W(g,J) = \sum_{\{I\}} \frac{1}{E!}(2P-1)!!\frac{(2P)!!(2P-3V)!}{(2P)!S_I}$$

From the definition of the double factorial, we have:

$$exp\ W(g,J) = \sum_{\{I\}} \frac{(2P)!}{E!}\frac{(2P-3V)!}{(2P)!S_I}$$

which gives:

$$exp\ W(g,J) = \sum_{\{I\}} \frac{1}{S_I}$$

Now we're ready to add in the cross-diagram contributions, as promised. We start with the assumption that $exp\ W(g,J)$ is the sum of all general diagrams (in fact it isn't really an assumption as much as the whole idea of a series expansion), and follow the derivation of equation 9.14. The result is:

$$exp\ W(g,J) = exp\left(\sum_{\{I\}} \frac{1}{S_I}\right)$$

which gives:

$$W(g,J) = \sum_{\{I\}} \frac{1}{S_I}$$

At this point we're summing over all the connected diagrams and determining everything from there. But we already know that we're going to get a factor of g^V and J^E, the only thing we actually need the diagram for is the symmetry factors. So let's factor:

$$W(g,J) = \sum_{V=0}^{\infty}\sum_{E=0}^{\infty}\left(\sum_{I} \frac{1}{S_I}\right) g^V J^E$$

which is equation 9.29.

Note: I don't know what the hell Srednicki's solution is supposed to prove. He claims that this "follows directly" from the phi-cubed theory, but that is obviously not true – the business with the propagator-vertex pairing, for example, is new and needs to be treated carefully. He also gives a downright incorrect explanation for why the propagator gets assigned a value of one in the Feynman diagrams. Moreover, this problem is in itself merely a calculus problem, and should be formulated completely independently of our QFT framework.

(b) Use equations 9.27 and 9.28 to compute $\mathbf{C_{V,E}}$ for $\mathbf{V \leq 4}$ and $\mathbf{E \leq 5}$ Verify that the symmetry factors given in figures 9.1-9.11 satisfy the sum rule of equation 9.29.

We go back to equation (9.4.1) (these follow from equation 9.27 and 9.28, so it's OK to start here):

$$exp\ W(g,J) = \sum_{V=0}^{4} \sum_{\substack{E=0 \\ E+3V \text{ even}}}^{5} \frac{1}{6^V V! E!}(3V+E-1)!! J^E g^V$$

We'll just bang these out:

$$exp\ W(g,J) = 1 + \left[\frac{5}{24}g^2 + \frac{385}{1152}g^4\right] + \left[\frac{1}{2}g + \frac{35}{48}g^3\right] J + \left[\frac{1}{2} + \frac{35}{48}g^2 + \frac{5005}{2304}g^4\right] J^2 +$$

$$\left[\frac{5}{12}g + \frac{385}{288}g^3\right] J^3 + \left[\frac{1}{8} + \frac{35}{64}g^2 + \frac{25025}{9216}g^4\right] J^4 + \left[\frac{7}{48}g + \frac{1001}{1152}g^3\right] J^4 + \ldots$$

Hence,

$$W(g,J) = log\left\{1 + \left[\frac{5}{24}g^2 + \frac{385}{1152}g^4\right] + \left[\frac{1}{2}g + \frac{35}{48}g^3\right] J + \left[\frac{1}{2} + \frac{35}{48}g^2 + \frac{5005}{2304}g^4\right] J^2 +\right.$$

$$\left.\left[\frac{5}{12}g + \frac{385}{288}g^3\right] J^3 + \left[\frac{1}{8} + \frac{35}{64}g^2 + \frac{25025}{9216}g^4\right] J^4 + \left[\frac{7}{48}g + \frac{1001}{1152}g^3\right] J^5 + \ldots\right\}$$

Now recall that $log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$. Expanding this is easy on Mathematica:

$$W(g,J) = \left(\frac{5}{24}g^2 + \frac{5}{16}g^4\right) + \left(\frac{1}{2}g + \frac{5}{8}g^3\right) J + \left(\frac{1}{2} + \frac{1}{2}g^2 + \frac{25}{16}g^4\right) J^2$$

$$+ \left(\frac{1}{6}g + \frac{2}{3}g^3\right) J^3 + \left(\frac{1}{8}g^2 + g^4\right) J^4 + \left(\frac{1}{8}g^3\right) J^5 + \ldots \qquad (9.4.2)$$

These clearly correspond to the symmetry factors presented in the textbook, according to the relation found in part (a).

(c) Now consider W(g,J+Y), with Y fixed by the "no tadpole" condition

$$\left.\frac{\partial}{\partial \mathbf{J}} \mathbf{W}(\mathbf{g}, \mathbf{J}+\mathbf{Y})\right|_{\mathbf{J}=\mathbf{0}} = \mathbf{0}$$

Then write

$$\mathbf{W}(\mathbf{g}, \mathbf{J}+\mathbf{Y}) = \sum_{\mathbf{V}=\mathbf{0}}^{\infty} \sum_{\mathbf{E}=\mathbf{0}}^{\infty} \widetilde{\mathbf{C}}_{\mathbf{V},\mathbf{E}} \mathbf{g}^{\mathbf{V}} \mathbf{J}^{\mathbf{E}}$$

Show that

$$\widetilde{\mathbf{C}}_{\mathbf{V},\mathbf{E}} = \sum_{\mathbf{I}} \frac{\mathbf{1}}{\mathbf{S}_{\mathbf{I}}}$$

where the sum is over all connected Feynman diagrams with E sources and V three-point vertices and no tadpoles, and $\mathbf{S_I}$ is the symmetry factor of each diagram.

There is nothing to show, Srednicki already proved this in the text. To summarize: we want our vacuum expectation value to be zero, so the sum of all one-source drawings must be zero. This source can be replaced with any other diagram, and the sum will still be zero. So, the counterterm forces the sum of the tadpoles to be zero. We enforce this in our expression by summing over everything except the tadpoles. We then repeat the argument of part (a) – absolutely nothing changes except that our sum is over all diagrams excluding tadpoles.

(d) Let $\mathbf{Y = a_1 g + a_3 g^3 + \ldots}$ and use equation 9.30 to determine $\mathbf{a_1}$ and $\mathbf{a_3}$. Compute $\mathbf{C_{V,E}}$ for $\mathbf{V \leq 4}$ and $\mathbf{E \leq 4}$. Verify that the symmetry factors for the diagrams in figure 9.13 satisfy the sum rule of part (c).

The condition is:

$$\frac{\partial}{\partial J} W(g, J+Y)\bigg|_{J=0} = 0$$

Taking the derivative, we arrive at:

$$\frac{1}{2}g + \frac{5}{8}g^3 + \left(1 + g^2 + \frac{50}{16}g^4\right)\left(a_1 g + a_3 g^3 + \ldots\right) + \left(\frac{1}{2}g + 2g^2\right)\left(a_1 g + a_3 g^3 + \ldots\right)^2$$

$$+ \left(\frac{1}{2}g^2 + 4g^4\right)\left(a_1 g + a_3 g^3\right)^3 + \ldots$$

The order g terms must be equal to zero, so:

$$\frac{1}{2}g + a_1 g = 0 \implies a_1 = -\frac{1}{2}$$

The order g^3 terms must also be equal to zero, so:

$$\frac{5}{8}g^3 + a_3 g^3 + a_1 g^3 + a_1^2 \frac{1}{2} g^3 = 0 \implies a_3 = -\frac{1}{4}$$

Using equation (9.4.2) with $J = J - \frac{1}{2}g - \frac{1}{4}g^3$, we have

$$W(g, J+Y) = \left(\frac{5}{24}g^2 + \frac{5}{16}g^4\right) + \left(\frac{1}{2}g + \frac{5}{8}g^3\right)\left(J - \frac{1}{2}g - \frac{1}{4}g^3\right)$$

$$+ \left(\frac{1}{2} + \frac{1}{2}g^2 + \frac{25}{16}g^4\right)\left(J - \frac{1}{2}g - \frac{1}{4}g^3\right)^2 + \left(\frac{1}{6}g + \frac{2}{3}g^3\right)\left(J - \frac{1}{2}g - \frac{1}{4}g^3\right)^3$$

$$+ \left(\frac{1}{8}g^2 + g^4\right)\left(J - \frac{1}{2}g - \frac{1}{4}g^3\right)^4 + \left(\frac{1}{8}g^3\right)\left(J - \frac{1}{2}g - \frac{1}{4}g^3\right)^5 + \ldots$$

Simplifying this on Mathematica is trivial, the result is:

$$W(g, J+Y) = \left(\frac{1}{12}g^2 + \frac{5}{48}g^4\right) + \left(\frac{1}{2} + \frac{1}{4}g^2 + \frac{5}{8}g^4\right)J^2 + \left(\frac{1}{6}g + \frac{5}{12}g^3\right)J^3 + \left(\frac{1}{8}g^2 + \frac{11}{16}g^4\right)J^4 + \ldots$$

These clearly correspond to the symmetry factors presented in the textbook, according to the relation found in part (c).

Srednicki 9.5. The interaction picture. In this problem, we will derive a formula for $\langle \mathbf{0} | \mathbf{T}\phi(\mathbf{x_n}) \ldots \phi(\mathbf{x_1}) | \mathbf{0} \rangle$ without using path integrals. Suppose we have a Hamiltonian density $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$, where $\mathcal{H}_0 = \frac{1}{2}\mathbf{\Pi^2} + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\mathbf{m^2}\phi^2$, and $\mathcal{H}_1$

is a function of $\mathbf{\Pi(x,0)}$ and $\boldsymbol{\phi}\mathbf{(x,0)}$ and their spatial derivatives. (It should be chosen to preserve Lorentz invariance, but we will not be concerned with this issue.) We will add a constant to H so that $\mathbf{H|0\rangle = 0}$. Let $\emptyset$ be the ground state of H_0, with a constant added to $\mathbf{H_0}$ so that $\mathbf{H_0|\emptyset\rangle = 0}$. ($H_1$ is then defined as H - H_0). The Heisenberg-picture field is:

$$\boldsymbol{\phi}\mathbf{(x,t) = e^{iHt}}\boldsymbol{\phi}\mathbf{(x,0)e^{-iHt}}$$

We now define the interaction picture field

$$\boldsymbol{\phi}_{\mathbf{I}}\mathbf{(x,t) = e^{iH_0t}}\boldsymbol{\phi}\mathbf{(x,0)e^{-iH_0t}}$$

(a) Show that $\boldsymbol{\phi}_{\mathbf{I}}\mathbf{(x)}$ obeys the Klein-Gordon equation and hence is a free field.

We have:

$$\phi_I = e^{iH_0t}\phi_S e^{-iH_0t}$$

where we define $\phi_S = \phi(x,0)$. Now we use Campbell-Baker-Hausdorff, and achieve:

$$\phi_I = \phi_S + [H_0t, \phi_S] + \frac{1}{2!}[H_0t, [H_0t, \phi_S]] + \frac{1}{3!}[H_0t, [H_0t, [H_0t, \phi_S]]] + \dots$$

which is:

$$\phi_I = \phi_S + t[H_0, \phi_S] + \frac{1}{2!}t^2[H_0, [H_0, \phi_S]] + \frac{1}{3!}t^3[H_0, [H_0, [H_0, \phi_S]]] + \dots \tag{9.5.1}$$

Now we use Srednicki 3.30:

$$H_0 = \int \widetilde{dk}\omega a^\dagger(k)a(k)$$

with ω defined in the usual way ($\omega = \sqrt{k^2+m^2}$). Just for future reference, let's do one calculation:

$$[H_0, a(k)] = \int \widetilde{dk'}\omega(k')\left[a^\dagger(k')a(k'), a(k)\right]$$

$$[H_0, a(k)] = \int \widetilde{dk'}i\omega(k')a(k')(2\pi)^3(2\omega)\delta^3(k-k')$$

$$[H_0, a(k)] = -i\omega(k)a(k)$$

Since they are at equal times,

$$[H_0, a(k)] = -i\omega a(k)$$

Similarly,

$$[H_0, a^\dagger(k)] = i\omega a^\dagger(k)$$

Now we're ready to evaluate:

$$[H_0, \phi_S] = \int \widetilde{dk}\left[H_0, a(k)e^{ik\cdot x} + a^\dagger(k)e^{-ik\cdot x}\right]$$

Note that there is no time-dependence here, since $t = 0$.

$$[H_0, \phi_S] = \int \widetilde{dk}\left(\left[H_0, a(k)\right]e^{ik\cdot x} + \left[H_0, a^\dagger(k)\right]e^{-ik\cdot x}\right)$$

$$[H_0, \phi_S] = \int \widetilde{dk} \left(-i\omega a(k)e^{ik\cdot x} + i\omega a^\dagger(k)e^{-ik\cdot x}\right)$$

Now plugging into equation (9.5.1), we have:

$$\phi_I = \int \widetilde{dk} \left[e^{ik\cdot x}e^{-i\omega t}a(k) + e^{-ik\cdot x}e^{i\omega t}a^\dagger(k)\right]$$

$$\phi_I = \int \widetilde{dk} \left[e^{ikx}a(k) + e^{-ikx}a^\dagger(k)\right]$$

Taking the spatial and temporal derivatives, and remembering that $kx = -\omega t + k \cdot x$, we find that:

$$(-\partial^2 + m^2)\phi_I = 0$$

which is the Klein-Gordon equation.

Note: Our fundamental axiom of quantum field theory is that bosons follow the Klein-Gordon equation. Assuming that our boson operator ϕ can be written in the form indicated, it is not necessary to do any math to show that the K-G equation is followed (the math will at some level hinge on the axiom, and the argument will therefore be circular). In this case, all the math was necessary merely to demonstrate that Srednicki's proposed interaction-picture operator is equivalent to the Schrödinger-operator usually used.

Nonetheless, this is a useful demonstration of the Campbell-Baker-Hausdorff theorem, and Srednicki's non-answer in the solution is a disappointing failure to address an interesting problem.

(b) Show that $\boldsymbol{\phi}(\mathbf{x}, \mathbf{t}) = \mathbf{U}^\dagger(\mathbf{t})\phi_\mathbf{I}(\mathbf{x})\mathbf{U}(\mathbf{t})$, where $\mathbf{U}(\mathbf{t}) = \mathbf{e}^{\mathbf{iH_0t}}\mathbf{e}^{-\mathbf{iHt}}$.

From equation 9.33,

$$\phi(x, t) = e^{iHt}\phi(x, 0)e^{-iHt}$$

Next, invert equation 9.34 to solve for $\phi(x, 0)$ and insert the result:

$$\phi(x, t) = e^{iHt}e^{-iH_0t}\phi_I(x, t)e^{iH_0t}e^{-iHt}$$

which is

$$\phi(x, t) = U^\dagger(t)\phi_I(x)U(t)$$

where $U(t) = e^{iH_0t}e^{-iHt}$

(c) Show that $\mathbf{U}(\mathbf{t})$obeys the differential equation $\mathbf{i}\frac{\mathbf{d}}{\mathbf{dt}}\mathbf{U}(\mathbf{t}) = \mathbf{H_I}(\mathbf{t})\mathbf{U}(\mathbf{t})$, where $\mathbf{H_I}(\mathbf{t}) = \mathbf{e}^{\mathbf{iH_0t}}\mathbf{H_1}\mathbf{e}^{-\mathbf{iH_0t}}$ is the interaction hamiltonian in the interaction picture, and the boundary condition $\mathbf{U}(\mathbf{0}) = \mathbf{1}$.

We have:

$$i\frac{d}{dt}U(t) = i\frac{d}{dt}e^{iH_0t}e^{-iHt}$$

$$i\frac{d}{dt}U(t) = \left(e^{iH_0t}(H)e^{-iHt} - (H_0)e^{iH_0t}e^{-iHt}\right)$$

$$i\frac{d}{dt}U(t) = e^{iH_0t}\left((H) - (H_0)\right)e^{-iHt}$$

$$i\frac{d}{dt}U(t) = e^{iH_0t}H_1e^{-iHt}$$

$$i\frac{d}{dt}U(t) = e^{iH_0t}H_1e^{-iH_0t}e^{iH_0t}e^{-iHt}$$

$$i\frac{d}{dt}U(t) = H_IU(t)$$

The boundary condition is obviously satisfied, since $e^0 = 1$.

(d) If $\mathcal{H}_1$ is specified by a particular function of the Schrödinger-picture fields $\Pi(\mathbf{x}, 0)$ and $\phi(\mathbf{x}, 0)$, show that $\mathcal{H}_I(\mathbf{t})$ is given by the same function of the interaction-picture fields $\Pi_I(\mathbf{x}, \mathbf{t})$ and $\phi_I(\mathbf{x}, \mathbf{t})$.

Whatever this function is, it can be expanded like this:

$$\mathcal{H}_I = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} C_{i,j}\Pi(x)^i\phi(x)^j$$

Time-evolving these states:

$$\mathcal{H}_I(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} C_{i,j}e^{iH_0t}\Pi(x)^i\phi(x)^je^{-iH_0t}$$

Now we just insert $i + j$ copies of the identity $e^{iH_0t}e^{-iH_0t}$. Then,

$$\mathcal{H}_I(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} C_{i,j}\Pi(x,t)^i\phi(x,t)^j$$

(e) Show that, for $\mathbf{t} \geq \mathbf{0}$.

$$\mathbf{U(t) = T\ exp}\left[-\mathbf{i}\int_{\mathbf{0}}^{\mathbf{t}}\mathbf{dt'H_I(t')}\right]$$

obeys the differential equation and boundary condition of part (c). What is the comparable expression for $\mathbf{t} \leq \mathbf{0}$? You may need to define a new ordering symbol.

This is trivial: simply take the derivative, apply the time-ordering symbol, and the differential equation is immediately satisfied. The boundary condition is also immediately satisfied. If you're concerned about how to take the derivative with respect to a variable specified only in the limits of integration, you should review the Fundamental Theorem of Calculus.

The time-ordering symbol may seem unnecessary, but remember that the integral can be split up into the sum of many smaller integrals, which then become a product of many exponentials. The derivative would act only on one exponential, so the $-iH_I$ from the derivative could go anywhere. Only the time-ordering symbol can restore order.

As for $t \leq 0$, the issue is that time t will now be the earliest t, but the differential equation requires the $-iH_I$ resulting from the derivative to be at the left of the equation. So, the expression will be the same as in eq. 9.35 for $t \leq 0$, but the time-ordering operator T must be replaced by an anti-time-ordering operator.

(f) Define $\mathbf{U(t_2, t_1) = U(t_2)U^\dagger(t_1)}$. Show that, for $\mathbf{t_2 > t_1}$,

$$\mathbf{U(t_2, t_1) = T\ exp\left[-i\int_{t_1}^{t_2} dt' H_I(t')\right]}$$

What is the comparable expression for $\mathbf{t_1 \geq t_2}$?

By definition:

$$U(t_2) = T\ exp\left[-i\int_0^{t_2} dt' H_I(t')\right]$$
$$U(t_1) = T\ exp\left[-i\int_0^{t_1} dt' H_I(t')\right]$$

Now we need to take the Hermitian conjugate of the second term. Recall that $(AB)^\dagger = B^\dagger A^\dagger$, ie the time-ordering is reversed. Then

$$U(t_2, t_1) = T\ exp\left[-i\int_0^{t_2} dt' H_I(t')\right] \overline{T}\ exp\left[i\int_0^{t_1} dt' H_I(t')\right]$$

Let's divide this first term:

$$U(t_2, t_1) = T\ exp\left[-i\int_{t_1}^{t_2} dt' H_I(t')\right] exp\left[-i\int_0^{t_1} dt' H_I(t')\right] \overline{T}\ exp\left[i\int_0^{t_1} dt' H_I(t')\right] \quad (9.5.2)$$

These last two terms cancel out. To do this in a formally correct way, let me separate out these last two terms:

$$T\ exp\left[-i\int_0^{t_1} dt' H_I(t')\right] \overline{T}\ exp\left[i\int_0^{t_1} dt' H_I(t')\right]$$

And expanding:

$$T\ exp\left[-i\int_\delta^{t_1} dt' H_I(t')\right] exp\left[-i\int_0^{\delta} dt' H_I(t')\right] \overline{T}\ exp\left[i\int_0^{\delta} dt' H_I(t')\right] exp\left[i\int_\delta^{t_1} dt' H_I(t')\right]$$

Now these middle terms cancel. These last two equations combined therefore show:

$$T\ exp\left[-i\int_0^{t_1} dt' H_I(t')\right] \overline{T}\ exp\left[i\int_0^{t_1} dt' H_I(t')\right] = T\ exp\left[-i\int_\delta^{t_1} dt' H_I(t')\right] \overline{T}\ exp\left[i\int_\delta^{t_1} dt' H_I(t')\right]$$

Repeating this an infinite number of times, the right hand side will eventually reach zero. Hence, equation (9.5.2) becomes:

$$U(t_2, t_1) = T\ exp\left[-i\int_{t_1}^{t_2} dt' H_I(t')\right]$$

As expected. As for the comparable expression for $t_1 > t_2$, we use the same setup. Note that both t_1 and t_2 can be defined to be positive, so the only thing that changes is how we divide the integral. The analog to equation (9.5.2) is:

$$U(t_2, t_1) = T\ exp\left[-i\int_0^{t_2} dt' H_I(t')\right] \overline{T}\ exp\left[i\int_0^{t_2} dt' H_I(t')\right] exp\left[i\int_{t_2}^{t_1} dt' H_I(t')\right]$$

So the result is the same except with the anti-time-ordering operator rather than the time-ordering operator.

(g) For any time ordering, show that $\mathbf{U(t_3, t_1) = U(t_3, t_2)U(t_2, t_1)}$ and that $\mathbf{U^\dagger(t_1, t_2) = U(t_2, t_1)}$.

By definition:

$$U^\dagger(t_1, t_2) = U(t_2)U^\dagger(t_1)$$

which is:

$$U^\dagger(t_1, t_2) = U(t_2, t_1)$$

□

Again by definition:

$$U(t_3, t_1) = U(t_3)U^\dagger(t_1)$$

By the definition of the unitary operator, we can insert an identity:

$$U(t_3, t_1) = U(t_3)U^\dagger(t_2)U(t_2)U^\dagger(t_1)$$

$$U(t_3, t_1) = U(t_3, t_2)U(t_2, t_1)$$

□

It is of course also possible to do this proof using the method of part (f), but that must be repeated for the different permutations of time-ordering.

(h) Show that

$$\boldsymbol{\phi}(\mathbf{x_n}) \dots \boldsymbol{\phi}(\mathbf{x_1}) = \mathbf{U^\dagger(t_n, 0)}\boldsymbol{\phi}_\mathbf{I}(\mathbf{x_n})\mathbf{U(t_n, t_{n-1})}\boldsymbol{\phi}_\mathbf{I}(\mathbf{x_{n-1}}) \dots \mathbf{U(t_2, t_1)}\boldsymbol{\phi}_\mathbf{I}(\mathbf{x_1})\mathbf{U(t_1, 0)}$$

Let's consider the right hand side:

$$U^\dagger(t_n, 0)\phi_I(x_n)U(t_n, t_{n-1})\phi_I(x_{n-1}) \dots U(t_2, t_1)\phi_I(x_1)U(t_1, 0)$$

Using the second equality in part (g):

$$\implies U(0, t_n)\phi_I(x_n)U(t_n, t_{n-1})\phi_I(x_{n-1}) \dots U(t_2, t_1)\phi_I(x_1)U(t_1, 0)$$

We've shown that U(0) = 0, so:

$$\implies U^\dagger(t_n)\phi_I(x_n)U(t_n)U^\dagger(t_{n-1})\phi_I(x_{n-1}) \dots U(t_2)U^\dagger(t_1)\phi_I(x_1)U(t_1)$$

Using the result of part (b) gives

$$\implies \phi(x_n) \dots \phi(x_1)$$

as expected. Was it OK that we used the results of part (b)? We've shown that the U of part (b) obeys the same differential equation with the same boundary conditions as the U of part (e). It turns out that ordinary differential equations have unique solutions when the function ($H_I(t)U(t)$, in our case) and its partial derivative (wrt t, in our case) are continuous. Hence, yes, the all the U(t)'s in this problem are equal.

(i) Show that $\mathbf{U^\dagger(t_n, 0) = U^\dagger(\infty, 0)U(\infty, t_n)}$ and also that $\mathbf{U(t_1, 0) = U(t_1, -\infty)U(-\infty, 0)}$.

This is just an application of part (g). Don't be concerned about the infinities: the unitary operators still work the same way, and are still defined since U(t) will have an improper (infinite) integral, which is well-defined for any reasonable boundary condition.

(j) Replace $\mathbf{H_0}$ with $\mathbf{(1 - i\epsilon)H_0}$, and show that $\mathbf{\langle 0|U^\dagger(\infty, 0) = \langle 0|\emptyset\rangle\langle\emptyset|}$ and that $\mathbf{U(-\infty, 0)|0\rangle = |\emptyset\rangle\langle\emptyset|0\rangle}$.

We have:

$$U(-\infty, 0)|0\rangle = U(-\infty)U^\dagger(0)|0\rangle$$

U(0) = 1, so:

$$U(-\infty, 0)|0\rangle = U(-\infty)|0\rangle$$

Using the results of part b:

$$U(-\infty, 0)|0\rangle = e^{-iH_0\infty}e^{iH\infty}|0\rangle$$

Replacing H_0 with $(1 - i\epsilon)H_0$ as instructed:

$$U(-\infty, 0)|0\rangle = e^{-i(1-i\epsilon)H_0\infty}e^{iH\infty}|0\rangle$$

The terms of the form $e^{ix}|0\rangle = |0\rangle$, because time-evolving the vacuum doesn't do anything. Hence,

$$U(-\infty, 0)|0\rangle = e^{-i(1-i\epsilon)H_0\infty}|0\rangle$$

Now the problem is that $|0\rangle$ is an eigenstate of the H operator, but we're dealing with the H_0 operator. Let's switch $|0\rangle$ to the H_0 eigenstate basis. Then:

$$U(-\infty, 0)|0\rangle = \sum_n e^{-i(1-i\epsilon)H_0\infty}|n\rangle\langle n|0\rangle$$

All the states except the vacuum are now multiplied by a factor of $e^{-\infty}$, and go to zero (for any $\epsilon > 0$, even a tiny one). Then,

$$U(-\infty, 0)|0\rangle = e^{-i(1-i\epsilon)H_0\infty}|\emptyset\rangle\langle\emptyset|0\rangle$$

The vacuum is completely unperturbed by the exponential, because time-evolving the vacuum doesn't do much. So,

$$U(-\infty, 0)|0\rangle = |\emptyset\rangle\langle\emptyset|0\rangle$$

as expected. Taking the Hermitian conjugate of this gives the other equality expected.

(k) Show that

$$\mathbf{\langle 0|\phi(x_n)\dots\phi(x_1)|0\rangle = \langle\emptyset|U(\infty, t_n)\phi_I(x_n)U(t_n, t_{n-1})\phi_I(x_{n-1})}$$

$$\mathbf{\dots U(t_2, t_1)\phi_I(x_1)U(t_1, -\infty)|\emptyset\rangle \times |\langle\emptyset|0\rangle|^2}$$

We use the result from part (h):

$$\phi(x_n)\dots\phi(x_1) = U^\dagger(t_n, 0)\phi_I(x_n)U(t_n, t_{n-1})\phi_I(x_{n-1})\dots U(t_2, t_1)\phi_I(x_1)U(t_1, 0)$$

Inserting the vacuum bra and ket:

$$\langle 0|\phi(x_n)\dots\phi(x_1)|0\rangle = \langle 0|U^\dagger(t_n,0)\phi_I(x_n)U(t_n,t_{n-1})\phi_I(x_{n-1})\dots U(t_2,t_1)\phi_I(x_1)U(t_1,0)|0\rangle$$

Now we use the result of part (i):

$$\langle 0|\phi(x_n)\dots\phi(x_1)|0\rangle = \langle 0|U^\dagger(\infty,0)U(\infty,t_n)\phi_I(x_n)U(t_n,t_{n-1})\phi_I(x_{n-1})$$
$$\dots U(t_2,t_1)\phi_I(x_1)U(t_1,-\infty)U(-\infty,0)|0\rangle$$

And finally, use the result of part (j):

$$\langle 0|\phi(x_n)\dots\phi(x_1)|0\rangle = \langle\emptyset|U(\infty,t_n)\phi_I(x_n)U(t_n,t_{n-1})\phi_I(x_{n-1})$$
$$\dots U(t_2,t_1)\phi_I(x_1)U(t_1,-\infty)|\emptyset\rangle \times |\langle 0|\emptyset\rangle|^2$$

(l) Show that

$$\boldsymbol{\langle 0|T\phi(x_n)\dots\phi(x_1)|0\rangle = \langle\emptyset|T\phi_I(x_n)\dots\phi_I(x_1)e^{-i\int d^4x\mathcal{H}_I(x)}|\emptyset\rangle \times |\langle\emptyset|0\rangle|^2}$$

We simply insert the time-ordering operator into the result of part (k). All the terms in the middle are of the form $U^\dagger(t_i)\phi(x_i)U(t_i)$: since these are all at equal time, the unitary operators don't do anything, and can be neglected. The result is:

$$\langle 0|T\ \phi(x_n)\dots\phi(x_1)|0\rangle = \langle\emptyset|T\ U(\infty)\phi_I(x_n)\dots\phi_I(x_1)U^\dagger(-\infty)|\emptyset\rangle \times |\langle\emptyset|0\rangle|^2$$

Now the time-ordering operator is still there, so we can write the remaining terms in whatever order we want – they'll be returned to their proper places by the T operator. We choose to move the first unitary operator to be near the second unitary operator. This gives the desired result:

$$\langle 0|T\phi(x_n)\dots\phi(x_1)|0\rangle = \langle\emptyset|T\phi_I(x_n)\dots\phi_I(x_1)e^{-i\int d^4x\mathcal{H}_I(x)}|\emptyset\rangle \times |\langle\emptyset|0\rangle|^2$$

(m) Show that

$$\boldsymbol{|\langle\emptyset|0\rangle|^2 = 1/\langle\emptyset|Te^{-i\int d^4x\mathcal{H}_i(x)}|\emptyset\rangle}$$

Thus we have

$$\boldsymbol{\langle 0|T\phi(x_n)\dots\phi(x_1)|0\rangle = \frac{\langle\emptyset|T\phi_I(x_n)\dots\phi_I(x_1)e^{-i\int d^4x\mathcal{H}_I(x)}|\emptyset\rangle}{\langle\emptyset|Te^{-i\int d^4x\mathcal{H}_I(x)}|\emptyset\rangle}}$$

We can now Taylor expand the exponentials on the right-hand side of equation 9.41, and use free-field theory to compute the resulting correlations functions.

This follows from part (l), simply set $\phi_i = 1$, and remember that $\langle 0|0\rangle = 1$.