Srednicki Chapter 8

QFT Problems & Solutions

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Srednicki 8.1. Starting with equation 8.11, verify equation 8.12

This is simple enough; we'll just plug 8.11 into the left side of equation 8.12:

$$(-\partial_x^2 + m^2)\Delta(x-x') = -\partial_x^2 \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-x')}}{k^2+m^2-i\varepsilon} + m^2 \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-x')}}{k^2+m^2-i\varepsilon}$$

which is:

$$(-\partial_x^2 + m^2)\Delta(x-x') = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')} \frac{k^2+m^2}{k^2+m^2-i\varepsilon}$$

Now let $\varepsilon \to 0$:

$$(-\partial_x^2 + m^2)\Delta(x-x') = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')}$$

Now use equation 3.27, only in four dimensions:

$$(-\partial_x^2 + m^2)\Delta(x-x') = \delta^4(x-x')$$

which is Srednicki 8.12.

Srednicki 8.2. Starting with equation 8.11, verify equation 8.13

Equation 8.11 is:

$$\Delta(x-x') = \frac{1}{(2\pi)^4} \int d^4k \frac{e^{ik(x-x')}}{k^2+m^2-i\varepsilon}$$

which is:

$$\Delta(x-x') = \frac{1}{(2\pi)^4} \int d^3k e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int d\omega \frac{e^{-i\omega(t-t')}}{-\omega^2+|\vec{k}|^2+m^2-i\varepsilon}$$

Now let $C^2 = |\vec{k}|^2 + m^2$:

$$\Delta(x-x') = \frac{1}{(2\pi)^4} \int d^3k e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int d\omega \frac{e^{-i\omega(t-t')}}{-\omega^2+C^2-i\varepsilon}$$

which is:

$$\Delta(x-x') = \frac{1}{(2\pi)^4} \int d^3k e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega(t-t')}}{(\omega-\sqrt{C^2-i\varepsilon})(-\omega-\sqrt{C^2-i\varepsilon})}$$

Now we note that an integral around the lower complex half plane will integrate to zero for $t-t' \geq 0$, since the exponential will then give $e^{-\infty}=0$ as the radius of the integral diverges. Then:

$$\Delta(x-x') = \frac{1}{(2\pi)^4}\int d^3k e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\int_C d\omega \frac{e^{-i\omega(t-t')}}{(\omega-\sqrt{C^2-i\varepsilon})(-\omega-\sqrt{C^2-i\varepsilon})}$$

We solve this integral with the residue theorem:

$$\Delta(x-x') = \frac{1}{(2\pi)^4}\int d^3k e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\,(-2\pi i)\text{Res}\left[\frac{e^{-i\omega(t-t')}}{(\omega-\sqrt{C^2-i\varepsilon})(-\omega-\sqrt{C^2-i\varepsilon})}\right]$$

which is:

$$\Delta(x-x') = -\frac{i}{(2\pi)^3}\int d^3k e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\,\text{Res}_{lower}\left[\frac{e^{-i\omega(t-t')}}{(\omega-\sqrt{C^2-i\varepsilon})(-\omega-\sqrt{C^2-i\varepsilon})}\right]$$

Our integral is is the lower half plane, so the relavant singularity in our integral is $\omega=+\sqrt{C^2-i\varepsilon}$. Then:

$$\Delta(x-x') = -\frac{i}{(2\pi)^3}\int d^3k e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\,\left[\frac{e^{-i\sqrt{C^2-i\varepsilon}(t-t')}}{(-2\sqrt{C^2-i\varepsilon})}\right]$$

which is:

$$\Delta(x-x') = i\int \frac{d^3k}{2(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\,\left[\frac{e^{-i\sqrt{C^2-i\varepsilon}(t-t')}}{(\sqrt{C^2-i\varepsilon})}\right]$$

Now we let $\varepsilon\to 0$:

$$\Delta(x-x') = i\int \frac{d^3k}{2(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\,\left[\frac{e^{-iC(t-t')}}{C}\right]$$

Note that with the $\varepsilon\to 0$, we have $C=\omega=\sqrt{|\vec{k}|^2+m^2}$. Technically it would be correct to eliminate ω completely and write our answer in terms of $\sqrt{|\vec{k}|^2+m^2}$, but it is more convenient to write ω and remember that $\omega=\sqrt{|\vec{k}|^2+m^2}$. In fact, we don't even need to "remember" it: notice that this condition can be rewritten as $-\omega^2+|\vec{k}|^2=-m^2$, which is just the on-shell condition for a massive particle (it's remarkable that we derived this condition from a seemingly unrelated problem involving the ground state of the free-field theory. This is a reassuring demonstration of the internal consistency of quantum field theory). Then:

$$\Delta(x-x') = i\int \frac{d^3k}{2(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\,\left[\frac{e^{-i\omega(t-t')}}{\omega}\right]$$

which is:

$$\Delta(x-x') = i\int \frac{d^3k}{2\omega(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}e^{-i\omega(t-t')}$$

As always, if the time ordering is reversed, then the integral must be taken in the upper half-plane, so the opposite singularity must be chosen, which will introduce a new negative sign in the definition of ω. Hence,

$$\Delta(x-x') = i\int \frac{d^3k}{2\omega(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}e^{i\omega(t-t')}$$

Combining these two results:

$$\Delta(x-x') = i\int \widetilde{dk} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}e^{-i\omega|t-t'|}$$

which is the first result of equation 8.13. We can simply rewrite this:

$$\Delta(x-x') = i\theta(t-t')\int \widetilde{dk} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}e^{-i\omega(t-t')} + i\theta(t'-t)\int \widetilde{dk} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}e^{i\omega(t-t')}$$

which is:

$$\Delta(x-x') = i\theta(t-t')\int \widetilde{dk} e^{ik(x-x')} + i\theta(t'-t)\int \widetilde{dk} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}e^{i\omega(t-t')}$$

We can simply negate the spatial k in the second term: this won't affect anything since we integrate over all of ks. Then,

$$\Delta(x-x') = i\theta(t-t')\int \widetilde{dk} e^{ik(x-x')} + i\theta(t'-t)\int \widetilde{dk} e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}e^{i\omega(t-t')}$$

This gives:

$$\Delta(x-x') = i\theta(t-t')\int \widetilde{dk} e^{ik(x-x')} + i\theta(t'-t)\int \widetilde{dk} e^{-ik(x-x')}$$

which is the second line of equation 8.13.

Srednicki 8.3. Starting with equation 8.13, verify equation 8.12.

Equation 8.13 gives:

$$(-\partial_x^2+m^2)\Delta(x-x') = (-\partial_x^2+m^2)\left(i\theta(t-t')\int \widetilde{dk} e^{ik(x-x')} + i\theta(t'-t)\int \widetilde{dk} e^{-ik(x-x')}\right)$$

which is:

$$(-\partial_x^2+m^2)\Delta(x-x') = -\partial_x^2\left(i\theta(t-t')\int \widetilde{dk} e^{ik(x-x')} + i\theta(t'-t)\int \widetilde{dk} e^{-ik(x-x')}\right)$$

$$+im^2\theta(t-t')\int \widetilde{dk} e^{ik(x-x')} + im^2\theta(t'-t)\int \widetilde{dk} e^{-ik(x-x')}$$

Evaluating this derivative, note that $\partial_x^2 = -\partial_t^2 + \partial_{\vec{x}}^2$. Then,

$$(-\partial_x^2+m^2)\Delta(x-x') = \partial_t^2\left(i\theta(t-t')\int \widetilde{dk} e^{ik(x-x')} + i\theta(t'-t)\int \widetilde{dk} e^{-ik(x-x')}\right)$$

$$-\partial_{\vec{x}}^2\left(i\theta(t-t')\int \widetilde{dk} e^{ik(x-x')} + i\theta(t'-t)\int \widetilde{dk} e^{-ik(x-x')}\right)$$

$$+im^2\theta(t-t')\int \widetilde{dk} e^{ik(x-x')} + im^2\theta(t'-t)\int \widetilde{dk} e^{-ik(x-x')}$$

which is:

$$(-\partial_x^2+m^2)\Delta(x-x') = \partial_t^2\left(i\theta(t-t')\int \widetilde{dk} e^{ik(x-x')} + i\theta(t'-t)\int \widetilde{dk} e^{-ik(x-x')}\right)$$

$$+\left(i\theta(t-t')\int \widetilde{dk}e^{ik(x-x')}(\vec{k}^2)+i\theta(t'-t)\int \widetilde{dk}e^{-ik(x-x')}(\vec{k}^2)\right)$$

$$+im^2\theta(t-t')\int \widetilde{dk}e^{ik(x-x')}+im^2\theta(t'-t)\int \widetilde{dk}e^{-ik(x-x')}$$

Distributing the derivatives, we get three terms from the first line:

$$(-\partial_x^2+m^2)\Delta(x-x')=i\delta'(t-t')\int \widetilde{dk}e^{ik(x-x')}-i\delta'(t-t')\int \widetilde{dk}e^{-ik(x-x')}$$

$$+2i\delta(t-t')\int \widetilde{dk}e^{ik(x-x')}(-i\omega)-2i\delta(t-t')\int \widetilde{dk}e^{-ik(x-x')}(-\omega)$$

$$+i\theta(t-t')\int \widetilde{dk}e^{ik(x-x')}(-\omega^2)+i\theta(t'-t)\int \widetilde{dk}e^{-ik(x-x')}(-\omega^2)$$

$$+i\theta(t-t')\int \widetilde{dk}e^{ik(x-x')}(\vec{k}^2)+i\theta(t'-t)\int \widetilde{dk}e^{-ik(x-x')}(\vec{k}^2)$$

$$+im^2\theta(t-t')\int \widetilde{dk}e^{ik(x-x')}+im^2\theta(t'-t)\int \widetilde{dk}e^{-ik(x-x')}$$

Note that in the first and second lines, the second term is negative because going from $\theta(-x)$ is negative at zero, so the delta function must be negative. The terms inside the delta function can then be negated at no cost before the second derivative is taken (this is a property of the delta function). Since the θ function jumps one unit, $1\delta(x-x')$ has the proper normalization.

The third, fourth, and fifth terms now vanish by the Klein-Gordon equation! Then,

$$(-\partial_x^2+m^2)\Delta(x-x')=i\delta'(t-t')\int \widetilde{dk}e^{ik(x-x')}-i\delta'(t-t')\int \widetilde{dk}e^{-ik(x-x')}$$

$$+2i\delta(t-t')\int \widetilde{dk}e^{ik(x-x')}(-i\omega)-2i\delta(t-t')\int \widetilde{dk}e^{-ik(x-x')}(-\omega)$$

In the first term, we integrate by parts:

$$(-\partial_x^2+m^2)\Delta(x-x')=-\delta(t-t')\int \widetilde{dk}e^{ik(x-x')}(\omega)-\delta(t-t')\int \widetilde{dk}e^{-ik(x-x')}(\omega)$$

$$+2\delta(t-t')\int \widetilde{dk}e^{ik(x-x')}(\omega)+2\delta(t-t')\int \widetilde{dk}e^{-ik(x-x')}(\omega)$$

This is:

$$(-\partial_x^2+m^2)\Delta(x-x')=\delta(t-t')\int \widetilde{dk}e^{ik(x-x')}(\omega)+\delta(t-t')\int \widetilde{dk}e^{-ik(x-x')}(\omega)$$

Now use equation 3.27, and remember the definition of $\widetilde{dk}$. We're still left with the $e^{i\omega(t-t')}$ parts of the exponentials, but the delta functions will render those unimportant as well. Then,

$$(-\partial_x^2+m^2)\Delta(x-x')=\delta(t-t')\frac{1}{2}\delta^3(\vec{x}-\vec{x}')+\delta(t-t')\frac{1}{2}\delta^3(\vec{x}-\vec{x}')$$

Negating the contents of the second delta function as before, these combine:

$$(-\partial_x^2+m^2)\Delta(x-x')=\delta(t-t')\delta^3(\vec{x}-\vec{x}')$$

which is:

$$(-\partial_x^2 + m^2)\Delta(x - x') = \delta^4(x - x')$$

as expected.

Srednicki 8.4. Use equations 3.19, 3.29, and 5.3 (and its Hermitian Conjugate) to verify the last line of equation 8.15.

Equation 3.19 implies:

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \langle 0|T \int \widetilde{dk_1}\widetilde{dk_2} \left[a(\vec{k_1})e^{ik_1x_1} + a^\dagger(\vec{k_1})e^{-ik_1x_1}\right] \left[a(\vec{k_2})e^{ik_2x_2} + a^\dagger(\vec{k_2})e^{-ik_2x_2}\right]|0\rangle$$

We'll assume these are time ordered as written. Then equation 5.3 gives:

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \langle 0| \int \widetilde{dk_1}\widetilde{dk_2} \left[a(\vec{k_1})e^{ik_1x_1}a^\dagger(\vec{k_2})e^{-ik_2x_2}\right]|0\rangle$$

This implies:

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \int \widetilde{dk_1}\widetilde{dk_2}e^{ik_1x_1-ik_2x_2}\langle 0|a(\vec{k_1})a^\dagger(\vec{k_2})|0\rangle$$

Now we use equation 3.29. This is a little bit complicated because the operators are not necessarily taken at the same time. Careful reading of the solution to problem 3.1 shows that we should actually replace the ω in this formula with $\frac{1}{2}(\omega_1 + \omega_2)$. However, k^2 is a constant, and so are the spatial parts (by the delta function that we're about to introduce), this subtetly will "come out in the wash," and it is easiest to simply use equation 3.29 as written, with $\omega = \omega_2$. Hence,

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \int \widetilde{dk_1}\widetilde{dk_2}e^{ik_1x_1-ik_2x_2}(2\pi)^3 2\omega_2\delta^3(\vec{k_1} - \vec{k_2})\langle 0|0\rangle$$

The bra-ket goes to one, leaving:

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \int \widetilde{dk_1}\widetilde{dk_2}e^{ik_1x_1-ik_2x_2}(2\pi)^3 2\omega_2\delta^3(\vec{k_1} - \vec{k_2})$$

Using the definition of $\widetilde{dk_2}$, we're left with:

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \int \widetilde{dk_1}d^3k_2e^{ik_1x_1-ik_2x_2}\delta^3(\vec{k_1} - \vec{k_2})$$

Doing the integral:

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \int \widetilde{dk_1}e^{ik_1(x_1-x_2)} \tag{8.4.1}$$

where the temporal parts are equal and therefore vanish, as mentioned above. Now we reverse the 1s and 2s, assuming the time ordering is opposite. This will introduce a negative sign. Combining these results:

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \theta(t_1 - t_2)\int \widetilde{dk}e^{ik(x_1-x_2)} + \theta(t_2 - t_1)\int \widetilde{dk}e^{-ik(x_1-x_2)}$$

which gives, using equation 8.13:

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \frac{1}{i}\Delta(x_2 - x_1)$$

which is the last line of equation 8.15.

Srednicki 8.5. The retarded and advanced Green's functions for the Klein-Gordon wave operator satisfy $\mathbf{\Delta_{ret}(x-y) = 0}$ for $\mathbf{x_0 \geq y_0}$ and $\mathbf{\Delta_{adv}(x-y) = 0}$ for $\mathbf{x_0 \leq y_0}$. Find the pole prescriptions on the right-hand side of eq. 8.11 that yield these Green's Functions.

$$\Delta(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2+m^2-i\epsilon}$$

Our goal is to set this equal to zero. To evaluate the time integral, we'll do add on an integral over the *lower* complex plane. This integral will go to zero, so it will not contribute to the integral (but it will allow us to use the Residue theorem). Now when we use the residue theorem, we'll get zero if both poles are in the upper complex plane.

Now, the only way this will happen is if we write Δ like this:

$$\Delta_{adv}(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{-(k^0)^2+\mathbf{k}^2+m^2-i\epsilon}$$

And choose Δ_{adv} to be like this:

$$\Delta_{adv}(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{-(k^0-i\epsilon)^2+\mathbf{k}^2+m^2}$$

This is the unique choice up to the coefficient of the ϵ, which doesn't matter anyway. Now we expand the binomial and throw away factors higher than first order in ϵ, leaving us with:

$$\Delta_{adv}(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2+m^2+2ik^0\epsilon}$$

Again, we don't care about the coefficient of the ϵ, so we'll drop the 2. We also don't care about the magnitude of the k^0, though we can't help keeping its sign. Then:

$$\Delta_{adv}(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2+m^2+i\ \mathrm{sign}(k^0)\epsilon}$$

The retarded Green's function makes us integrate over the upper half plane, so the singularities must be downstairs, and so:

$$\Delta_{ret}(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2+m^2-i\ \mathrm{sign}(k^0)\epsilon}$$

Srednicki 8.6. Let $\mathbf{Z_0(J) = \exp(iW_0(J))}$ and evaluate the real and imaginary parts of $\mathbf{W_0(J)}$.

From the first line of equation 8.10, we see that W_0 is defined as:

$$W_0(J) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{\widetilde{J}(k)\widetilde{J}(-k)}{k^2+m^2-i\epsilon}$$

We're done, up to separating the real and imaginary parts. Now, notice that $J(x)$ is real. Its Fourier transform is then follows

$$\widetilde{(J)}(k)^* = \int e^{-ikx}f(x)dx = \widetilde{J}(-k)$$

Thus,

$$W_0(J) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{\widetilde{J}(k)\widetilde{J}^*(k)}{k^2+m^2-i\epsilon}$$

which is:

$$W_0(J) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{|\widetilde{J}(k)|^2}{k^2+m^2-i\epsilon}$$

Now, everything is manifestly real up to the ϵs in the bottom. So, we multiply by the complex conjugate:

$$W_0(J) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{|\widetilde{J}(k)|^2}{k^2+m^2-i\epsilon}\cdot\frac{k^2+m^2+i\epsilon}{k^2+m^2+i\epsilon}$$

which is:

$$W_0(J) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}|\widetilde{J}(k)|^2\frac{k^2+m^2+i\epsilon}{(k^2+m^2)^2+\epsilon^2}$$

The real part is obviously:

$$\boxed{Re[W_0(J)] = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}|\widetilde{J}(k)|^2\frac{k^2+m^2}{(k^2+m^2)^2+\epsilon^2}}$$

Srednicki calls this fraction the "principle part" of $\frac{1}{k^2+m^2}$; however, I do not use this notation, as I find it unnecessarily cumbersome.

As for the imaginary part, we have:

$$Im[W_0(J)] = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}|\widetilde{J}(k)|^2\frac{\epsilon}{(k^2+m^2)^2+\epsilon^2}$$

Now, note that as $\varepsilon \to 0$, this will go to zero – unless k^2+m^2 is also zero, in which case the product goes to infinity. This is a delta function, which we write as:

$$Im[W_0(J)] = \frac{A}{2}\int \frac{d^4k}{(2\pi)^4}|\widetilde{J}(k)|^2\delta(k^2+m^2)$$

The normalization constant A is determined by integrating over all k^2+m^2 – it turns out that the integral is π as $\varepsilon \to 0$. Then,

$$\boxed{Im[W_0(J)] = \frac{\pi}{2}\int \frac{d^4k}{(2\pi)^4}|\widetilde{J}(k)|^2\delta(k^2+m^2)}$$

Srednicki 8.7. Repeat the analysis of this section for the complex scalar field that was introduced in problem 3.5 and further studied in problem 5.1.

(a) Write your source term in the form $\mathbf{J}^\dagger\boldsymbol{\phi} + \mathbf{J}\boldsymbol{\phi}^\dagger$, and find an explicit formula, analogous to equation 8.10, for $\mathbf{Z_0}(\mathbf{J}^\dagger, \mathbf{J})$.

Recall that the Lagrangian for this field is:

$$\mathcal{L} = -\partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi + \Omega_0$$

Though Srednicki is not clear about this, $\mathcal{L}_0$ is just the lagrangian without the last term, since the last term is a renormalization term rather than part of the density itself. Equation 8.3 therefore gives:

$$Z_0(J) = \int \mathcal{D}\phi exp \left[i \int d^4x \left\{ -\partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi + J^\dagger \phi + J\phi^\dagger \right\} \right]$$

The action is:

$$S_0(J) = \int d^4x \left\{ -\partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi + J^\dagger \phi + J\phi^\dagger \right\}$$

Recall that the Fourier Transform is

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{ik_\mu x^\mu} \widetilde{\phi}(k)$$

We take the derivative. Note that $\partial_\mu \phi = \frac{\partial}{\partial x^\mu}\phi$. Then:

$$\partial_\mu \phi = \int \frac{d^4k}{(2\pi)^4} (ik_\mu) e^{ik_\mu x^\mu} \widetilde{\phi}(k)$$

Using this, the first term in the action becomes:

$$-\int d^4x \partial^\mu \phi^\dagger \partial_\mu \phi = -\int d^4x \frac{d^4k}{(2\pi)^4} (ik^\mu) e^{-ikx} \widetilde{\phi}^\dagger(k) \frac{d^4k'}{(2\pi)^4} (-ik_\mu) e^{ik'x} \widetilde{\phi}(k')$$

$$-\int d^4x \partial^\mu \phi^\dagger \partial_\mu \phi = -\int d^4x \frac{d^4k}{(2\pi)^4} (k^\mu) e^{-i(k-k')x} \widetilde{\phi}^\dagger(k) \frac{d^4k'}{(2\pi)^4} (k_\mu) \widetilde{\phi}(k')$$

Doing the x-integral:

$$-\int d^4x \partial^\mu \phi^\dagger \partial_\mu \phi = -\int \frac{d^4k}{(2\pi)^4} k^2 \delta^4(k-k') \widetilde{\phi}^\dagger(k) d^4k' \widetilde{\phi}(k')$$

Doing the k' integral:

$$-\int d^4x \partial^\mu \phi^\dagger \partial_\mu \phi = -\int \frac{d^4k}{(2\pi)^4} \widetilde{\phi}^\dagger(k) k^2 \widetilde{\phi}(k)$$

The other three terms proceed similarly: we Fourier transform ϕ and J and do any easy integrals. The result is:

$$S_0 = \int \frac{d^4k}{(2\pi)^4} \left[-\widetilde{\phi}^\dagger(k)(k^2+m^2)\widetilde{\phi}(k) + \widetilde{J}^\dagger(k)\widetilde{\phi}(k) + \widetilde{\phi}^\dagger(k)\widetilde{J}(k) \right]$$

Now we define:

$$\widetilde{\chi} = \widetilde{\phi} - \frac{\widetilde{J}(k)}{k^2+m^2}$$

Now the action becomes:

$$S_0 = \int \frac{d^4k}{(2\pi)^4}\left[-\widetilde{\chi}^\dagger(k^2+m^2)\widetilde{\chi} + \frac{\widetilde{J}^\dagger(k)\widetilde{J}(k)}{k^2+m^2}\right]$$

Hence:

$$Z_0 = \int \mathcal{D}\widetilde{\chi} exp\left\{i\left[\int \frac{d^4k}{(2\pi)^4}\left(-\widetilde{\chi}^\dagger(k^2+m^2)\widetilde{\chi} + \frac{\widetilde{J}^\dagger(k)\widetilde{J}(k)}{k^2+m^2}\right)\right]\right\}$$

The χ dependence is exclusively in the first term, so the second term can move in front of the integral. The remaining integral, however, just gives the probability of a ground state to ground state transition without any external forces – which is obviously one. Hence, only the term in front of the integral survives:

$$Z_0 = exp\left\{i\int \frac{d^4k}{(2\pi)^4}\left(\frac{\widetilde{J}^\dagger(k)\widetilde{J}(k)}{k^2+m^2}\right)\right\}$$

Fourier transforming back:

$$Z_0 = exp\left\{i\left(\int d^4x d^4x' J^\dagger(x)\Delta(x-x')J(x')\right)\right\}$$

(b) Write down the appropriate generalization of equation 8.14, and use it to compute $\langle 0|T\phi(x_1)\phi(x_2)|0\rangle$, $\langle 0|T\phi^\dagger(x_1)\phi(x_2)|0\rangle$, and $\langle 0|T\phi^\dagger(x_1)\phi^\dagger(x_2)|0\rangle$.

These two fields are considered completely separate, so our analog to equation 8.14 is simply:

$$\langle 0|T\phi(x_1)\ldots\phi^\dagger(y_1)\ldots|0\rangle = \frac{1}{i}\frac{\delta}{\delta J(x_1)}\cdots\frac{1}{i}\frac{\delta}{\delta J^\dagger(y_1)}\cdots\Big|_{J=0,J^\dagger=0}$$

The first of these correlation functions is:

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = -\frac{\delta}{\delta J^\dagger(x_1)}\frac{\delta}{\delta J^\dagger(x_2)}exp\left(i\int d^4x d^4x' J^\dagger(x)\Delta(x-x')J(x')\right)\Big|_{J=0,J^\dagger=0}$$

Taking the first functional derivative (remember the rules for how to do this!), we obtain:

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = -\frac{\delta}{\delta J^\dagger(x_1)}\left(i\int d^4x d^4x' \delta^4(x-x_2)\Delta(x-x')J(x')\right)$$
$$exp\left(i\int d^4x d^4x' J^\dagger(x)\Delta(x-x')J(x')\right)\Big|_{J=0,J^\dagger=0}$$

We now do the x integral in the first term:

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = -i\frac{\delta}{\delta J^\dagger(x_1)}\left(\int d^4x' \Delta(x_2-x')J(x')\right)$$

$$exp\left(i\int d^4xd^4x'J^\dagger(x)\Delta(x-x')J(x')\right)\Bigg|_{J=0,J^\dagger=0} \tag{8.7.2}$$

The first term doesn't have any $J^\dagger$ terms left, so the derivative passes through:

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = -i\left(\int d^4x'\Delta(x_2-x')J(x')\right)\times$$

$$\frac{\delta}{\delta J^\dagger(x_1)}exp\left(i\int d^4xd^4x'J^\dagger(x)\Delta(x-x')J(x')\right)\Bigg|_{J=0,J^\dagger=0}$$

Doing this integral as well, we find:

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \left(\int d^4x'\Delta(x_2-x')J(x')\right)\left(\int d^4x'\Delta(x_1-x')J(x')\right)\times$$

$$exp\left(i\int d^4xd^4x'J^\dagger(x)\Delta(x-x')J(x')\right)\Bigg|_{J=0,J^\dagger=0}$$

We now set $J=0$:

$$\boxed{\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = 0}$$

Now we take the above argument and switch $J\leftrightarrow J^\dagger$, $\phi\leftrightarrow\phi^\dagger$. The argument still works; we're left with:

$$\boxed{\langle 0|T\phi^\dagger(x_1)\phi^\dagger(x_2)|0\rangle = 0}$$

To evaluate the remaining correlation function, we use the analog of equation (8.7.2):

$$\langle 0|T\phi^\dagger(x_1)\phi(x_2)|0\rangle = -i\frac{\delta}{\delta J(x_1)}\left(\int d^4x'\Delta(x_2-x')J(x')\right)$$

$$exp\left(i\int d^4xd^4x'J^\dagger(x)\Delta(x-x')J(x')\right)\Bigg|_{J=0,J^\dagger=0}$$

Since both of these terms have J terms, we have to use the product rule:

$$\langle 0|T\phi^\dagger(x_1)\phi(x_2)|0\rangle = -i\left[\left(\int d^4x'\Delta(x_2-x')\delta(x_1-x')\right)\right.$$

$$\left.exp\left(i\int d^4xd^4x'J^\dagger(x)\Delta(x-x')J(x')\right)\right] - i\left[\left(\int d^4x'\Delta(x_2-x')J(x')\right)\right.$$

$$\left.\left(i\int d^4xd^4x'J^\dagger(x)\Delta(x-x')\delta(x'-x_1)\right)exp\left(i\int d^4xd^4x'J^\dagger(x)\Delta(x-x')J(x')\right)\right]\Bigg|_{J=0,J^\dagger=0}$$

We can use these delta functions to do two integrals:

$$\langle 0|T\phi^\dagger(x_1)\phi(x_2)|0\rangle =$$

$$-i\left[\Delta(x_2-x_1)exp\left(i\int d^4xd^4x'J^\dagger(x)\Delta(x-x')J(x')\right)\right] + \left[\left(\int d^4x'\Delta(x_2-x')J(x')\right)\right.$$

$$\left.\left(\int d^4xJ^\dagger(x)\Delta(x-x_1)\right)exp\left(i\int d^4xd^4x'J^\dagger(x)\Delta(x-x')J(x')\right)\right]\Bigg|_{J=0,J^\dagger=0}$$

Setting J = 0, the second term vanishes, as does the argument in the exponential. Hence,

$$\boxed{\langle 0|T\phi^\dagger(x_1)\phi(x_2)|0\rangle = -i\Delta(x_2 - x_1)}$$

(c) Verify your results by using the method of problem 8.4

We use equation 3.38 to expand this result:

$$\langle 0|T\phi^\dagger(x_1)\phi(x_2)|0\rangle = \int \widetilde{dk}\widetilde{dk'}\langle 0| \left(a^\dagger(\vec{k})e^{-ikx} + b(\vec{k})e^{ikx}\right) \left(a(\vec{k'})e^{ik'x} + b^\dagger(\vec{k'})e^{-ik'x}\right)|0\rangle$$

Equation 5.3 gives:

$$\langle 0|T\phi^\dagger(x_1)\phi(x_2)|0\rangle = \int \widetilde{dk}\widetilde{dk'}e^{ikx-ik'x}\langle 0|b(\vec{k'})b^\dagger(\vec{k'})|0\rangle \tag{8.7.3}$$

Using 5.3 again and doing the integral, we have:

$$\langle 0|T\phi^\dagger(x_1)\phi(x_2)|0\rangle = \int \widetilde{dk}e^{ikx-ik'x}$$

This is equivalent to equation (8.4.1) in problem 8.4, and so the result is the same:

$$\langle 0|T\phi^\dagger(x_1)\phi(x_2)|0\rangle = -i\Delta(x_2 - x_1)$$

which is what we expected.

As for the zero results in part (b), we can immediately see that the analogs of equation (8.7.3) for these correlation functions are:

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle \propto \langle 1\phi|1\phi^\dagger\rangle$$

and

$$\langle 0|T\phi^\dagger(x_1)\phi^\dagger(x_2)|0\rangle \propto \langle 1\phi^\dagger|1\phi\rangle$$

both of which will be zero in a non-interacting theory, since there is a zero transition amplitude from a ϕ particle to a $\phi^\dagger$ particle.

(d) Finally, give the appropriate generalization of equation 8.17

Just as we saw in part b, for every ϕ that does not have a $\phi^\dagger$, there will be an extra J is the prefactor. For every $\phi^\dagger$ that does not have a ϕ, there were similarly be an extra $J^\dagger$. Both of these cause the correlation function to vanish. If the numbers are equal, we must pair up the functional derivatives in an appropriate way to get a nonzero result. Generally,

$$\langle 0|T\phi(x_1)\ldots\phi^\dagger(y_1)\ldots|0\rangle = \frac{1}{i^n}\sum_{\text{pairings}} \Delta(x_1 - y_1)\ldots\Delta(x_n - y_m)$$

where we pair up each x with each y. Note that we cannot pair the xs or ys with themselves, as this corresponds to two functional J or $J^\dagger$ derivatives, which will vanish when acting on $\exp\left[J^\dagger(x)\Delta(x-y)J(y)\right]$. Further, this result is defined to be zero if $n \neq m$.

Srednicki 8.8. A harmonic oscillator (in units with $m = \hbar = 1$) has a ground-state wave function $\langle \mathbf{q}|\mathbf{0}\rangle \propto \mathbf{e}^{-\omega \mathbf{q}^2/2}$. Now consider a real scalar field $\phi(\mathbf{x})$, and define a *field eigenstate* $|\mathbf{A}\rangle$ that obeys

$$\boldsymbol{\phi}(\mathbf{x}, \mathbf{0})|\mathbf{A}\rangle = \mathbf{A}(\mathbf{x})|\mathbf{A}\rangle$$

where the function $\mathbf{A}(\mathbf{x})$ is everywhere real. For a free-field theory specified by the Hamiltonian of equation 8.1, show that the ground state wave function is:

$$\langle \mathbf{A}|\mathbf{0}\rangle \propto \mathbf{exp}\left[-\frac{\mathbf{1}}{\mathbf{2}}\int \frac{\mathbf{d^3k}}{(\mathbf{2}\pi)^\mathbf{3}}\boldsymbol{\omega}(\mathbf{k})\widetilde{\mathbf{A}}(\mathbf{k})\widetilde{\mathbf{A}}(-\mathbf{k})\right]$$

where $\widetilde{\mathbf{A}}(\mathbf{k}) = \int \mathbf{d^3x e^{-ikx} A(x)}$ and $\boldsymbol{\omega}(\mathbf{k}) = \sqrt{\mathbf{k^2 + m^2}}$.

This problem is a bit opaque. The Hamiltonian given is:

$$\mathcal{H} = \frac{1}{2}\left[\Pi^2 + (\nabla\phi)^2 + m^2\phi^2\right]$$

Let's Fourier transform this, using equation 8.6 (in only three dimensions). The Hermitian conjugate terms will be interpreted to have negative momentum as usual. Thus,

$$\widetilde{\mathcal{H}} = \int \frac{d^3k}{2(2\pi)^3}\left[\widetilde{\Pi}(k)\widetilde{\Pi}(-k) + (\hat{p}^2 + m^2)\widetilde{\phi}(k)\widetilde{\phi}(-k)\right]$$

Acting with the momentum operator, we have:

$$\widetilde{\mathcal{H}} = \frac{1}{(2\pi)^3}\int d^3k\frac{1}{2}\left[\widetilde{\Pi}(k)\widetilde{\Pi}(-k) + (\vec{k}^2 + m^2)\widetilde{\phi}(k)\widetilde{\phi}(-k)\right]$$

Using the definition of ω:

$$\widetilde{\mathcal{H}} = \frac{1}{(2\pi)^3}\int d^3k\frac{1}{2}\left[\widetilde{\Pi}(k)\widetilde{\Pi}(-k) + \omega^2\widetilde{\phi}(k)\widetilde{\phi}(-k)\right]$$

But under the integral, this is exactly the form of equation 7.1! In other words, our free field is just a collection of an infinite number of harmonic oscillators. We must therefore determine the wave functions for all our k's and multiply them together. We will therefore have an infinite product of terms of the form $e^{-\omega\widetilde{A}^2/2}$ (note that q is the eigenstate of the position operator in quantum mechanics; so similarly, here A is the eigenstate of the field operator, as in eq. 8.18).

How to convert our integral (infinite sum) into an infinite product? Note that an infinite product of exponentials is equivalent to one exponential of an infinite sum. Hence, our wave function is proportional to:

$$\langle A|0\rangle = exp\left[-\frac{1}{2(2\pi)^3}\int d^3k\omega(k)\widetilde{A}(k)\widetilde{A}(-k)\right]$$

If you're wondering where the $(2\pi)^3$ came from, equation 7.1's definition of ω, as well as that of the problem statement, is equal to our ω divided by $(2\pi)^3$ (to see this, just match up the terms in our Hamiltonian expression with those of equation 7.1).