Srednicki Chapter 70

QFT Problems & Solutions

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Srednicki 70.1. Verify equation 70.10.

The discussion directly after equation 70.9 is:

$$T_R^a T_R^a = C(R)\delta^{aa} \tag{70.1.1}$$

Taking the trace:

$$\mathrm{Tr}(T_R^a T_R^a) = C(R)\mathrm{Tr}(\delta^{aa})$$

Now equation (70.1.1) tells us that that δ function must have the same dimensionality as the representation, ie D(R). Thus:

$$\mathrm{Tr}(T_R^a T_R^a) = C(R)D(R)$$

Now let's use equation 70.9:

$$T(R)\mathrm{Tr}(\delta^{ab}) = C(R)D(R)$$

Now consider equation 70.9: we're summing over the as, ie over the number of generators, ie from 1 to D(A). The trace of this must therefore be D(A). This gives:

$$T(R)D(A) = C(R)D(R)$$

Srednicki 70.2. (a) Use equations 70.12 and 70.26 to compute T(A) for SU(N)

We use equation 70.12:

$$T(N \otimes \overline{N}) = T(1 \oplus A)$$

Now we use 70.12:

$$T(N \oplus \overline{N}) = T(1) + T(A)$$

Next comes 70.15:

$$T(N)D(\overline{N}) + T(\overline{N})D(N) = T(1) + T(A)$$

The dimensionality of N (or $\overline{N}$) is obviously just N, so:

$$T(N)N + T(\overline{N})N = T(1) + T(A)$$

It is stated below equation 70.10 that $T(N) = T(\overline{N}) = \frac{1}{2}$ for SU(N), so:

$$N = T(1) + T(A)$$

Now T(1) is the index for a one-dimensional representation, which has no commutation relations. Therefore $T(1) = 0$, and:

$$T(A) = N$$

(b) For SU(2), the adjoint representation is specified by $(\boldsymbol{T_A})^{\boldsymbol{bc}} = \boldsymbol{-i\varepsilon^{abc}}$. Use this to compute $\boldsymbol{T(A)}$ explicitly for SU(2). Does your result agree with part (a)?

We have, using 70.9:

$$Tr\left[(T_A^a)^{cd}(T_A^b)^{de}\right] = T(A)\delta^{ab}$$

Using the adjoint representation given in the problem:

$$Tr\left[-\varepsilon^{acd}\varepsilon^{bde}\right] = T(A)\delta^{ab}$$

Now we multiply by δ^{ab}:

$$Tr\left[\varepsilon^{adc}\varepsilon^{ade}\right] = T(A)\delta^{aa}$$

Now a is on both sides, so it is fixed once and for all. The other two can go sequentially or anti-sequentially from the first; the results will be $(1)^2$ or $(-1)^2$ respectively, and so:

$$Tr\left[2\right] = T(A)$$

where the delta function on the right hand side is gone, since the one value of a will always equal itself. Then:

$$T(A) = 2$$

(c) Consider the SU(2) subgroup of SU(N) that acts on the first two components of the fundamental representation of SU(N). Under this SU(2) subgroup, the N of SU(N) transforms as $\boldsymbol{2 \oplus (N-2)1_s}$. Using equation 70.26, figure out how the adjoint representation of SU(N) transforms under this SU(2) subgroup.

As instructed, we use equation 70.26:

$$N \otimes \overline{N} = 1 \oplus A$$

Now we transform N:

$$[2 \oplus (N-2)1_S] \otimes [2 \oplus (N-2)1_s] = 1 \oplus A$$

This gives:

$$(2 \otimes 2) \oplus [(N-2)1_S \otimes 2] \oplus [2 \otimes (N-2)1_S] \oplus (N-2)^2(1_S \otimes 1_s) = 1 \oplus A \qquad (70.2.1)$$

which gives:

$$(3 \oplus 1) \oplus (2N-4)2_S \oplus (N-2)^2 1_S = 1 \oplus A$$

Thus:

$$A = 3 \oplus (2N-4)2_S \oplus (N-2)^2 1_S$$

(d) Use your results from parts (b) and (c) to compute T(A) for SU(N). Does your result agree with part (a)?

We use the result from part (c):

$$T(A) = T(3) + (2N-4)T(2_S) + (N-2)^2 T(1_S)$$

Now $2 \otimes 2 = 1 + 3_S$, where 3 is the adjoint representation. Thus T(3) = T(A) = 3 in SU(2). Further, T(1) is 0, because a one-dimensional matrix does not have any commutation relations. Finally, T(2) we know from 70.6 is 1/2. Thus:

$$T(A) = 2 + (2N-4)\frac{1}{2} + (N-2)^2 0$$

Thus:

$$T(A) = N$$

which is the result from part (a).

Srednicki 70.3. (a) Consider the SO(3) subgroup of SO(N) that acts on the first three components of the fundamental representations of SO(N). Under this SO(3) subgroup, the N of SO(N) transforms as $\mathbf{3 \oplus (N-3)1_s}$. Using equation 70.29, work out how the adjoint representation of SO(N) transforms under this SO(3) subgroup.

We use equation 70.29:

$$N \otimes \overline{N} = 1_S \oplus A_A \oplus S_S$$

Now we transform N:

$$[3 \oplus (N-3)1_s] \otimes [3 \oplus (N-3)1_s] = 1_S \oplus A_A \oplus S_S$$

This gives:

$$(3 \otimes 3) \oplus (N-3)(3 \otimes 1_S) \oplus (N-3)(1_S \otimes 3) \oplus (N-3)^2 1_s = 1_S \oplus A_A \oplus S_S$$

To evaluate these group multiplication elements, we use Young Tableaux, as described in Sakurai section 6.5. For $3 \otimes 3$, we have:

$$3 \otimes 3 = \begin{array}{|c|c|}\hline & \\ \hline \end{array} \oplus \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$$

where this first term has $\begin{array}{|c|c|}\hline 0 & 0 \\ \hline \end{array}$, $\begin{array}{|c|c|}\hline 0 & 1 \\ \hline \end{array}$, $\begin{array}{|c|c|}\hline 0 & 2 \\ \hline \end{array}$, $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}$, $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}$, $\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \end{array}$, and the second term has: $\begin{array}{|c|}\hline 0 \\ \hline 1 \\ \hline \end{array}$, $\begin{array}{|c|}\hline 0 \\ \hline 2 \\ \hline \end{array}$, $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$. This means that $3 \otimes 3 = 6_S \oplus 3_A$.

Similarly, we get $3 \otimes 1 = 2_S + 1_A$. Now we drop the symmetric terms and we have:

$$3_A \oplus (N-3)2_A \oplus (N-3)1_A = A$$

This gives:

$$A = (N-3)3_A \oplus (1)3_A$$

And so:

$$A = (N-2)3_A$$

(b) Use the resuts from part (a) and from problem 70.2 to compute T(A) for SO(N).

We use the result from part (a):

$$T(A) = (N-2)T(3_A)$$

We found in problem 70.2 that T(3) = 2. Then:

$$T(A) = 2N - 4$$

Srednicki 70.4. (a) For SU(N), we have:

$$\boldsymbol{N} \otimes \boldsymbol{N} = \boldsymbol{\mathcal{A}_A} + \boldsymbol{\mathcal{S}_S}$$

where $\boldsymbol{\mathcal{A}}$ corresponds to a field with two antisymetric fundamental SU(N) indices $\boldsymbol{\phi_{ij} = -\phi_{ji}}$, and $\boldsymbol{\mathcal{S}}$ corresponds to a field with two symmetric fundamental SU(N) indices $\boldsymbol{\phi_{ij} = \phi_{ji}}$. Compute $\boldsymbol{D(\mathcal{A})}$ and $\boldsymbol{D(\mathcal{S})}$.

This is just a matter of degrees of freedom (D represents the dimension, ie the number of generators, ie the number of degrees of freedom). An NxN matrix that is symmetric has $(N^2+N)/2$ degrees of freedom, (half the non-diagonal elements, and the diagonal) while the antisymmetric matrix has $(N^2-N)/2$ degrees of freedom (half the non-diagonal elements). Thus, we have:

$$D(\mathcal{A}) = \frac{1}{2}(N^2 - N)$$

$$D(\mathcal{S}) = \frac{1}{2}(N^2 + N)$$

(b) By considering an SU(2) subgroup of SU(N), compute T(A) and T(S).

Using equation (70.2.1) from problem 70.2, we have:

$$(2 \otimes 2) \oplus [(N-2)1_S \otimes 2] \oplus [2 \otimes (N-2)1_S] \oplus (N-2)^2(1_S \otimes 1_s) = 1 \oplus A$$

Using the equation in the problem statement, we have:

$$(2 \otimes 2) \oplus [(N-2)1_S \otimes 2] \oplus [2 \otimes (N-2)1_S] \oplus (N-2)^2(1_S \otimes 1_s) = 1 \oplus A_A + S_S$$

Now we take the anti-symmetric component of this.

$$1_A \oplus (N-2)2_S = A_A$$

And the symmetric component:

$$3_S \oplus (N-2)2_S \oplus (N-2)^2 1_S = S_S \oplus 1 \tag{70.2.2}$$

Now we take the index of both sides:

$$T(A) = T(1_A) \oplus (N-2)T(2_S)$$

Now we know 1_A is invariant, and $T(2_S) = \frac{1}{2}$. Then:

$$T(A) = \frac{1}{2}(N-2)$$

Similarly, we take the index on equation (70.2.2):

$$T(S) + T(1) = T(3_S) \oplus (N-2)T(2_S) \oplus (N-2)^2 T(1_S)$$

Again, we have $T(2) = \frac{1}{2}$ and $T(1) = 0$. We also found in problem 70.2(d) that $T(3) = 2$. Then:

$$T(S) = 2 + \frac{1}{2}(N-2)$$

which is:

$$T(S) = \frac{1}{2}(N+2)$$

(c) For SU(3), show that $\boldsymbol{\mathcal{A}} = \overline{\mathbf{3}}$.

We summarize the desired properties with:

$$\phi_{ij} = \varepsilon_{ijk}\phi^k$$

This is the representation for an antisymmetric triplet.

(d) By considering an SU(3) subgroup of SU(N), compute A(A) and A(S).

We begin with:

$$A_A + S_S = N \otimes N$$

From the statement of problem 70.3, we transform the SU(3) component of SU(N) such that:

$$A_A + S_S = [3 \oplus (N-3)1_s] \otimes [3 \oplus (N-3)1_S]$$

Distributing:

$$A_A + S_S = [3 \otimes 3] \oplus [3 \otimes (N-3)1_s] \oplus [(N-3)1_S \otimes 3] \oplus (N-3)^2[1_S \otimes 1_S] \qquad (70.2.3)$$

Taking the anti-symmetric components of this:

$$A = \overline{3} + (N-3)3_S$$

Now we take the anomaly coefficient such that:

$$A(A) = A(\overline{3}) + (N-3)A(3)$$

using the discussion at the bottom of equation 427, we have:

$$A(A) = -1 + (N-3) \cdot 1$$

which is:

$$\boxed{A(A) = N - 4}$$

Now we use equation (70.2.3):

$$S = 6 + (N-3)3 + (N-3)^2 1$$

Taking the anomaly coefficient:

$$A(S) = A(6) + (N-3)A(3) + (N-3)^2 A(1)$$

Now A(3) = 1 and A(1) = 0, and so:

$$A(S) = A(6) + (N-3) \tag{70.2.4}$$

Now we just need A(6). We have from equation 70.36:

$$A(3 \otimes 3) = A(3)D(3) + D(3)A(3)$$

which is:

$$A(6 \oplus \overline{3}) = 1 \cdot 3 + 3 \cdot 1$$

where A(3) is the same as before, and D(3) is given on page 426. Now we use 70.35 again:

$$A(6) + A(\overline{3}) = 6$$

Which gives:

$$A(6) = 7$$

Putting this into equation (70.2.4):

$$\boxed{A(S) = N + 4}$$

Note: Are you confused about why $[3 \otimes (N-3)1] + [(N-3)1 \otimes 3]$ *is half symmetric and half anti-symmetric? So am I! It's clearly true – consider the SU(3) component of SU(4) – but I can't seem to prove it. In fact, I'm really very frustrated about this kind of ambiguity and the lack of clear examples in this section in the text. It would be very helpful to include a brute-force calculation with several possible representations for one group like SU(3). Perhaps I will write an extra supplement on this topic after I understand it better.*

Srednicki 70.5. Consider a field ϕ_i in the representation R_1 and a field χ_I in the representation R_2. Their product $\phi_i X_I$ is then in the direct product representation $R_1 \otimes R_2$, with generator matrices given by equation 70.13.

(a) Prove the distribution rule for the covariant derivative,

$$[D_\mu(\phi\chi)]_{iI} = (D_\mu\phi)_i\chi_I + \phi_i(D_\mu\chi)_I$$

I hate this notation, because the indices are very ambiguous. What's really going on is that the covariant derivative (D) has two indices, and the fields have one each. This is a little opaque because the fields are in different representations, and so cannot be directly multiplied. As a result, the covariant derivative really has four indices, two in each reprentation. I find it less confusing (though a little misleading) to write the index directly on the fields, and leave the covariant derivative's indices implied (I say misleading just because the written index will actually contract with the index on the covariant derivative).

In any case, the discussion on page 420 and the partial derivatives give:

$$D_\mu(\phi_i\chi_I) = \partial_\mu(\phi_i\chi_I) - igA^a_\mu(T^a_{R_1\oplus R_2})_{iI,jJ}\phi_j\chi_J$$

Using the product rule and equation 70.13:

$$D_\mu(\phi_i\chi_I) = \phi_i\partial_\mu\chi_I + (\partial_\mu\phi_i)\chi_I - igA^a_\mu(T^a_{R1})_{ij}\delta_{IJ}\phi_j\chi_J - igA^a_\mu\delta_{ij}(T^a_{R2})_{IJ}\phi_j\chi_J$$

Using the delta functions and reordering:

$$D_\mu(\phi_i\chi_I) = \phi_i\left[\partial_\mu\chi_I - igA^a_\mu(T^a_{R2})_{IJ}\chi_J\right] + \left[\partial_\mu\phi_i - igA^a_\mu(T^a_{R1})_{ij}\phi_j\right]\chi_I$$

which is:

$$D_\mu(\phi_i\chi_I) = \phi_i(D_\mu\chi_I) + (D_\mu\phi_I)\chi_I$$

(b) Consider a field $\boldsymbol{\phi_i}$ in the complex representation R. Show that

$$\boldsymbol{\partial_\mu(\phi^{\dagger i}\phi_i) = (D_\mu\phi^\dagger)^i\phi_i + \phi^{\dagger i}(D_\mu\phi_I)}$$

We have on page 424 that $\phi^{\dagger i}\phi_i$ is invariant, ie a singlet, ie that $T^a_1 = 0$. Thus, the partial and covariant derivatives are equivalent, and our result from part (a) gives us the desired result.

Srednicki 70.6. The field strength in Yang-Mills theory is in the adjoint representation, and so its covariant derivative is:

$$\boldsymbol{(D_\rho F_{\mu\nu})^a = \partial_\rho F^a_{\mu\nu} - igA^c_\rho(T^c_A)^{ab}F^b_{\mu\nu}}$$

Prove the *Bianchi Identity*:

$$\boldsymbol{(D_\mu F_{\nu\rho})^a + (D_\nu F_{\rho\mu})^a + (D_\rho F_{\mu\nu})^a = 0}$$

We use the definition of the covariant derivative:

$$(\partial_\mu F_{\nu\rho})^a + (\partial_\nu F_{\rho\mu})^a + (\partial_\rho F_{\mu\nu})^a - igA^c_\mu(T^c_A)^{ab}F^b_{\nu\rho} - igA^c_\nu(T^c_A)^{ab}F^b_{\rho\mu} - igA^c_\rho(T^c_A)^{ab}F^b_{\mu\nu} = 0$$

Now we use equation 70.3, which shows that $(T^a_A)^{bc} = -if^{abc}$:

$$(\partial_\mu F_{\nu\rho})^a + (\partial_\nu F_{\rho\mu})^a + (\partial_\rho F_{\mu\nu})^a - gA^c_\mu f^{cab}F^b_{\nu\rho} - gA^c_\nu f^{cab}F^b_{\rho\mu} - gA^c_\rho f^{cab}F^b_{\mu\nu} = 0$$

Next we use equation 69.22 to expand the field strength:

$$\partial_\mu\partial_\nu A^a_\rho - \partial_\mu\partial_\rho A^a_\nu + gf^{dea}\partial_\mu(A^d_\nu A^e_\rho) + \partial_\nu\partial_\rho A^a_\mu - \partial_\nu\partial_\mu A^a_\rho + gf^{dea}\partial_\nu(A^d_\rho A^e_\mu)$$

$$\partial_\rho \partial_\mu A^a_\nu - \partial_\rho \partial_\nu A^a_\mu - \partial_\rho \partial_\nu A^a_\mu + g f^{dea} \partial_\rho (A^d_\mu A^e_\nu) - g A^c_\mu f^{cab} \left[\partial_\nu A^b_\rho - \partial_\rho A^b_\nu + g f^{deb} A^d_\nu A^e_\rho \right]$$
$$-g A^c_\nu f^{cab} \left[\partial_\rho A^b_\mu - \partial_\mu A^b_\rho + g f^{deb} A^d_\rho A^e_\mu \right] - g A^c_\rho f^{cab} \left[\partial_\mu A^b_\nu - \partial_\nu A^b_\mu + g f^{deb} A^d_\mu A^e_\nu \right] = 0$$

Now we use the commutativity of partial derivatives to cancel the first and fifth terms the second and seventh, and the fourth and eighth. We further use the product rule, and reorder some terms:

$$g f^{dea} \left[A^d_\nu (\partial_\mu A^e_\rho) + (\partial_\mu A^d_\nu) A^e_\rho + A^d_\rho (\partial_\nu A^e_\mu) + (\partial_\nu A^d_\rho) A^e_\mu + A^d_\mu (\partial_\rho A^e_\nu) + (\partial_\rho A^d_\mu) A^e_\nu - A^e_\mu (\partial_\nu A^d_\rho) \right.$$
$$\left. + A^e_\mu (\partial_\rho A^d_\nu) - A^e_\nu (\partial_\rho A^d_\mu) + A^e_\nu (\partial_\mu A^d_\rho) - A^e_\rho (\partial_\mu A^d_\nu) + A^e_\rho (\partial_\nu A^d_\mu) \right] - g^2 f^{cab} f^{deb} \left[A^c_\mu A^d_\nu A^e_\rho + \right.$$
$$\left. A^c_\nu A^d_\rho A^e_\mu + A^c_\rho A^d_\mu A^e_\nu \right] = 0$$

The second and eleventh terms cancel, so do the fourth and seventh, as well as the sixth and ninth. Further, we do lots of distributing. This gives:

$$g \left[f^{dea} A^d_\nu (\partial_\mu A^e_\rho) + f^{dea} A^d_\rho (\partial_\nu A^e_\mu) + f^{dea} A^d_\mu (\partial_\rho A^e_\nu) + f^{dea} A^e_\mu (\partial_\rho A^d_\nu) + f^{dea} A^e_\nu (\partial_\mu A^d_\rho) + f^{dea} A^e_\rho (\partial_\nu A^d_\mu) \right]$$
$$-g^2 \left[f^{cab} f^{deb} A^c_\mu A^d_\nu A^e_\rho + f^{cab} f^{deb} A^e_\mu A^c_\nu A^d_\rho + f^{cab} f^{deb} A^d_\mu A^e_\nu A^c_\rho \right] = 0$$

Now we take in the first part $d, e \to b, c$ just for convenience. We further change a few of the dummy indices to promote factoring or cancellation. We also divide by g:

$$f^{bca} A^b_\nu (\partial_\mu A^c_\rho) + f^{cba} A^b_\nu (\partial_\mu A^c_\rho) + f^{bca} A^b_\rho (\partial_\nu A^c_\mu) + f^{cba} A^b_\rho (\partial_\nu A^c_\mu) + f^{bca} A^b_\mu (\partial_\rho A^c_\nu) + f^{cba} A^b_\mu (\partial_\rho A^c_\nu)$$
$$-g A^c_\mu A^d_\nu A^e_\rho \left[f^{cab} f^{deb} + f^{dab} f^{ecb} + f^{eab} f^{cdb} \right] = 0$$

Now we use the antisymmetry of f to cancel these first terms. We further divide by -g. This gives:

$$A^c_\mu A^d_\nu A^e_\rho \left[f^{acb} f^{bed} + f^{adb} f^{bce} + f^{eab} f^{cdb} \right] = 0$$

Just for our sanity, let's change these indices to Srednicki's notation: ($c \to b$, $b \to d$, $e \to c$, $d \to e$). This gives:

$$A^b_\mu A^e_\nu A^c_\rho \left[f^{abd} f^{dce} + f^{aed} f^{dbc} + f^{cad} f^{bed} \right] = 0$$

Now we use the antisymmetry of f:

$$A^b_\mu A^e_\nu A^c_\rho \left[f^{abd} f^{dce} + f^{bcd} f^{dae} + f^{cad} f^{dbe} \right] = 0$$

which vanishes by the Jacobi Identity, proving the Bianchi Identity.