Srednicki Chapter 7

QFT Problems & Solutions

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Srednicki 7.1. Starting with equation 7.12, do the contour integral to verify equation 7.14

Equation 7.12 is:

$$G(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dE \frac{e^{-iE(t-t')}}{-E^2+\omega^2-i\varepsilon}$$

Now we can write this as:

$$G(t-t') = -\frac{1}{2\pi}\int_{-\infty}^{\infty} dE e^{-iE(t-t')} \frac{1}{E^2-(\omega^2-i\varepsilon)}$$

Hence:

$$G(t-t') = -\frac{1}{4\pi\sqrt{\omega^2-i\varepsilon}}\int_{-\infty}^{\infty} dE e^{-iE(t-t')} \left[\frac{1}{E-\sqrt{\omega^2-i\varepsilon}} - \frac{1}{E+\sqrt{\omega^2-i\varepsilon}}\right] \tag{7.1.1}$$

Now let's assume that $t-t' \geq 0$. In this case, an integral over a semi-circle the lower half-plane will go to zero, since the exponential factor will have a term of (-i) as written, a term proportional to (-i) from the circle in the lower half-plane, and a positive term from the (t - t'). The result is $e^{-\text{positive real number * mod(E)}}$ which will go to zero as $|E| \to \infty$. As a result, we can tack on the integral over the semi-circle (since it equals zero), ending up with a contour integral:

$$G(t-t') = -\frac{1}{4\pi\sqrt{\omega^2-i\varepsilon}}\int_{C} dE e^{-iE(t-t')} \left[\frac{1}{E-\sqrt{\omega^2-i\varepsilon}} - \frac{1}{E+\sqrt{\omega^2-i\varepsilon}}\right]$$

Now the residue theorem tells us that we can take $-2\pi i$ times the residue, where the residue is just for those singularities in the lower half plane (that is to say, those with the negative imaginary part). Hence,

$$G(t-t') = -\frac{i}{2\sqrt{\omega^2-i\varepsilon}} \text{ Res } \frac{e^{-iE(t-t')}}{E-\sqrt{\omega^2-i\varepsilon}}$$

Now we let $E = \sqrt{\omega^2-i\varepsilon}$, which is the singularity. We drop the part that goes to zero (that's the beauty of taking the residue), and we're left with:

$$G(t-t') = \frac{i}{2\sqrt{\omega^2-i\varepsilon}} e^{-i\sqrt{\omega^2-i\varepsilon}(t-t')}$$

where the negative sign disappeared since we did the integral in the clockwise direction, and the definition of the residue is taken assuming a counterclockwise direction. (By the way, we know it's the clockwise direction because the real integral ends at positive infintity, and we need to make our way back to negative infinity by going through the lower complex plane. This is obviously clockwise).

Now we let $\varepsilon \to 0$:

$$G(t-t') = \frac{i}{2\omega}e^{-i\omega(t-t')} \text{ for t - t' } \geq 0$$

Good. Now let's assume that $t - t' \leq 0$. In this case we have to add on a semi-circle in the upper half-plane, and we need to use the other singularity (since that's the semi-circle that equals zero). Everything will work the same as before, except for an overall minus sign (since we're using the other singularity, which has a minus sign in front of it) – but even that is cancelled since the contour integral is in the counterclockwise direction this time. The only real difference is in the exponential, since the opposite singularity was chosen:

$$G(t-t') = \frac{i}{2\omega}e^{i\omega(t-t')} \text{ for t - t' } \leq 0$$

Combining these results, we have:

$$G(t-t') = \frac{i}{2\omega}e^{-i\omega|t-t'|}$$

which is equation 7.14.

Srednicki 7.2. Starting with equation 7.14, verify equation 7.13.

Equation 7.14 is:

$$G = \frac{i}{2\omega}e^{-i\omega|t-t'|}$$

Now we take the derivative:

$$\frac{\partial G}{\partial t} = \frac{i}{2\omega}(-i\omega)sign(t-t')e^{-i\omega|t-t'|}$$

$$\frac{\partial G}{\partial t} = \frac{sign(t-t')}{2}e^{-i\omega|t-t'|}$$

Now for another derivative:

$$\frac{\partial^2 G(t-t')}{\partial t^2} = \delta(t-t')e^{-i\omega|t-t'|} - i\omega\frac{sign(t-t')^2}{2}e^{-i\omega|t-t'|}$$

This first term is only nonzero if t = t', in which case the exponential doesn't contribute. Hence:

$$\frac{\partial^2 G(t-t')}{\partial t^2} = \delta(t-t') - i\omega\frac{sign(t-t')^2}{2}e^{-i\omega|t-t'|}$$

We'll also move the last term to the other side:

$$\frac{\partial^2 G(t-t')}{\partial t^2} + i\omega\frac{sign(t-t')^2}{2}e^{-i\omega|t-t'|} = \delta(t-t')$$

The sign function squared is one everywhere except at t = t'. So we'll specialize to the case where $t \neq t'$:

$$\frac{\partial^2 G(t-t')}{\partial t^2} + \frac{i\omega}{2} e^{-i\omega|t-t'|} = \delta(t-t')$$

which is:

$$\frac{\partial^2 G(t-t')}{\partial t^2} + \omega^2 G(t-t') = \delta(t-t')$$

This is equation 7.13.

What about if t = t'? As it turns out, this proof – and equation 7.13 itself – don't hold in that case! But that's okay because there's no need to do path integrals if t = t'! In that case, the amplitude is just one if the initial and final states are the same and zero otherwise.

Srednicki 7.3. (a) Use the Heisenberg equation of motion, $\dot{\mathbf{A}} = \mathbf{i}[\mathbf{H}, \mathbf{A}]$, to find explicit expressions for $\dot{\mathbf{Q}}$ and $\dot{\mathbf{P}}$. Solve these to get the Heisenberg-picture operators Q(t) and P(t) in terms of the Schrödinger-picture operators Q and P.

Using the formula for P:

$$\dot{P} = i[H, P]$$

$$\dot{P} = i[\frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 Q^2, P]$$

$$\dot{P} = \frac{i}{2}m\omega^2 [Q^2, P]$$

$$\dot{P} = \frac{i}{2}m\omega^2 \left(Q[Q,P] + [Q,P]Q\right)$$

$$\dot{P} = -m\omega^2 Q \tag{7.3.1}$$

Note that $\hbar$ was set to 1, but m was not (Srednicki set them both to one). Using the formula for Q:

$$\dot{Q} = i[H, Q]$$

$$\dot{Q} = i[\frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 Q^2, Q]$$

$$\dot{Q} = \frac{i}{2m}[P^2, Q]$$

$$\dot{Q} = \frac{i}{m}[P, Q]P$$

$$\dot{Q} = \frac{1}{m}P \tag{7.3.2}$$

Differentiating equation (7.3.1):

$$\ddot{P} = -m\omega^2 \dot{Q}$$

Inserting equation (7.3.2):

$$\ddot{P} = -\omega^2 P \tag{7.3.3}$$

Similarly, differentate equation (7.3.2):

$$\ddot{Q} = \frac{1}{m}\dot{P}$$

Inserting equation (7.3.1):

$$\ddot{Q} = -\omega^2 Q \tag{7.3.4}$$

Solving equtations (7.3.3) and (7.3.4), we have:

$$P(t) = Acos(\omega t) + Bsin(\omega t)$$

$$Q(t) = Ccos(\omega t) + Dsin(\omega t)$$

Now, at time 0, the Schrödinger and Heisenberg pictures are the same – the kets haven't evolved in the Schrödinger picture, and the operators haven't evolved in the Heisenberg Picture. Hence, $P(0) = P$ and $Q(0) = Q$, etc. So:

$$P(t) = Pcos(\omega t) + Bsin(\omega t)$$

$$Q(t) = Qcos(\omega t) + Dsin(\omega t)$$

Now we differentiate these and take the values at t = 0.

$$\dot{P}(0) = B\omega$$

$$\dot{Q}(0) = D\omega$$

Now we use equations (7.3.1) and (7.3.2), noting that the right hand side of these equations are the Schrödinger picture operators, since the operators haven't evolved yet at t = 0. Then,

$$\dot{P}(0) = B\omega = -m\omega^2 Q$$

$$\dot{Q}(0) = D\omega = \frac{1}{m}P$$

Hence,

$$B = -m\omega Q$$

$$D = \frac{1}{m\omega}P$$

and so:

$$P(t) = Pcos(\omega t) - m\omega Qsin(\omega t) \tag{7.3.5}$$

$$Q(t) = Qcos(\omega t) + \frac{1}{m\omega}Psin(\omega t) \tag{7.3.6}$$

(b) Write the Schrödinger-picture operators Q and P in terms of the creation and annihilation operators a and $\mathbf{a^\dagger}$, where $\mathbf{H = \hbar\omega(a^\dagger a + \frac{1}{2})}$. Then, using your result from part (a), write the Heisenberg-picture operators Q(t) and P(t) in terms of a and $\mathbf{a^\dagger}$.

This is actually not a derivation as much as a definition. We use the usual definition from quantum mechanics, see Sakurai 2.3.24:

$$Q = \sqrt{\frac{1}{2m\omega}}(a^\dagger + a)$$

$$P = i\sqrt{\frac{m\omega}{2}}(a^\dagger - a)$$

We know that this definition is okay because the consequent Hamiltonian (Sakurai 2.3.6) is equivalent to that in the problem statement.

Now we use equations (7.3.5) and (7.3.6):

$$P(t) = i\sqrt{\frac{m\omega}{2}}(a^\dagger - a)cos(\omega t) - \sqrt{\frac{m\omega}{2}}(a^\dagger + a)sin(\omega t)$$

$$Q(t) = \sqrt{\frac{1}{2m\omega}}(a^\dagger + a)cos(\omega t) + i\sqrt{\frac{1}{2m\omega}}(a^\dagger - a)sin(\omega t)$$

which is:

$$P(t) = i\sqrt{\frac{m\omega}{2}}\left[a^\dagger\left(cos(\omega t) + isin(\omega t)\right) - a\left(cos(\omega t) - isin(\omega t)\right)\right]$$

$$Q(t) = \sqrt{\frac{1}{2m\omega}}\left[a^\dagger\left(cos(\omega t) + isin(\omega t)\right) + a\left(cos(\omega t) - isin(\omega t)\right)\right]$$

Hence:

$$P(t) = i\sqrt{\frac{m\omega}{2}}\left[a^\dagger e^{i\omega t} - ae^{-i\omega t}\right] \tag{7.3.7}$$

$$Q(t) = \sqrt{\frac{1}{2m\omega}}\left[a^\dagger e^{i\omega t} + ae^{-i\omega t}\right] \tag{7.3.8}$$

(c) Using your result from part (b), and $\mathbf{a}|\mathbf{0}\rangle = \langle\mathbf{0}|\mathbf{a}^\dagger = \mathbf{0}$, verify equations 7.16 and 7.17.

Using equation (7.3.8):

$$\langle 0|TQ(t_1)Q(t_2)|0\rangle = \langle 0|T\sqrt{\frac{1}{2m\omega}}\left[a^\dagger e^{i\omega t} + ae^{-i\omega t}\right]\sqrt{\frac{1}{2m\omega}}\left[a^\dagger e^{i\omega t} + ae^{-i\omega t}\right]|0\rangle$$

which is:

$$\langle 0|TQ(t_1)Q(t_2)|0\rangle = \frac{1}{2m\omega}\langle 0|T\left[a^\dagger e^{i\omega t_1} + ae^{-i\omega t_1}\right]\left[a^\dagger e^{i\omega t_2} + ae^{-i\omega t_2}\right]|0\rangle$$

We'll assume that $t_1 \geq t_2$ Most of these terms annihilate the vacuum. The only one that doesn't is:

$$\langle 0|TQ(t_1)Q(t_2)|0\rangle = \frac{1}{2m\omega}e^{i\omega(t_2-t_1)}\langle 0|aa^\dagger|0\rangle$$

which is:

$$\langle 0|TQ(t_1)Q(t_2)|0\rangle = \frac{1}{2m\omega}e^{-i\omega(t_1-t_2)}$$

What about if $t_2 \geq t_1$? In this case, the time ordering is opposite, so we simply take $t_1 \leftrightarrow t_2$:

$$\langle 0|TQ(t_1)Q(t_2)|0\rangle = \frac{1}{2m\omega}e^{i\omega(t_1-t_2)}$$

Combining these we have:

$$\langle 0|TQ(t_1)Q(t_2)|0\rangle = \frac{1}{2m\omega}e^{-i\omega|t_1-t_2|}$$

which is:

$$\langle 0|TQ(t_1)Q(t_2)|0\rangle = \frac{1}{i}\frac{i}{2m\omega}e^{-i\omega|t_1-t_2|}$$

Using equation 7.14 (and noting that Srednicki sets $m=1$:

$$\langle 0|TQ(t_1)Q(t_2)|0\rangle = \frac{1}{i}G(t_2-t_1)$$

which is equation 7.16. □

As for 7.17, the procedure is the same. We start by assuming that $t_1 \geq t_2 \geq t_3 \geq t_4$:

$$\langle 0|TQ(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle = \langle 0|\sqrt{\frac{1}{2m\omega}}\left[a^\dagger e^{i\omega t} + ae^{-i\omega t}\right]\sqrt{\frac{1}{2m\omega}}\left[a^\dagger e^{i\omega t} + ae^{-i\omega t}\right]$$

$$\sqrt{\frac{1}{2m\omega}}\left[a^\dagger e^{i\omega t} + ae^{-i\omega t}\right]\sqrt{\frac{1}{2m\omega}}\left[a^\dagger e^{i\omega t} + ae^{-i\omega t}\right]|0\rangle$$

$$\langle 0|TQ(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle = \left(\frac{1}{2m\omega}\right)^2\langle 0|\left[a^\dagger e^{i\omega t_1} + ae^{-i\omega t_1}\right]\left[a^\dagger e^{i\omega t_2} + ae^{-i\omega t_2}\right]$$

$$\left[a^\dagger e^{i\omega t_3} + ae^{-i\omega t_3}\right]\left[a^\dagger e^{i\omega t_4} + ae^{-i\omega t_4}\right]|0\rangle$$

Some of these terms vanish:

$$\langle 0|TQ(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle = e^{i\omega(t_4-t_1)}\left(\frac{1}{2m\omega}\right)^2\langle 0|a\left[a^\dagger e^{i\omega t_2} + ae^{-i\omega t_2}\right]\left[a^\dagger e^{i\omega t_3} + ae^{-i\omega t_3}\right]a^\dagger|0\rangle$$

which gives:

$$\langle 0|TQ(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle = e^{i\omega(t_4-t_1)}\left(\frac{1}{2m\omega}\right)^2\langle 0|a\left[a^\dagger a^\dagger e^{i\omega(t_2+t_3)} + a^\dagger a e^{i\omega(t_2-t_3)}\right.$$

$$\left.+aa^\dagger e^{i\omega(t_3-t_2)} + aae^{-i\omega(t_2+t_3)}\right]a^\dagger|0\rangle$$

and then:

$$\langle 0|TQ(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle = e^{i\omega(t_4-t_1)}\left(\frac{1}{2m\omega}\right)^2\langle 0|aa^\dagger a^\dagger a^\dagger e^{i\omega(t_2+t_3)} + aa^\dagger aa^\dagger e^{i\omega(t_2-t_3)}$$

$$+aaa^\dagger a^\dagger e^{i\omega(t_3-t_2)} + aaaa^\dagger e^{-i\omega(t_2+t_3)}|0\rangle$$

The first and last terms eventually annihilate the vacuum:

$$\langle 0|TQ(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle = e^{i\omega(t_4-t_1)}\left(\frac{1}{2m\omega}\right)^2\langle 0|aa^\dagger aa^\dagger e^{i\omega(t_2-t_3)} + aaa^\dagger a^\dagger e^{i\omega(t_3-t_2)}|0\rangle$$

The operators now cancel out nicely, leaving:

$$\langle 0|TQ(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle = \left(\frac{1}{2m\omega}\right)^2\left(e^{i\omega(t_4-t_1)}e^{i\omega(t_2-t_3)} + 2e^{i\omega(t_4-t_1)}e^{i\omega(t_3-t_2)}\right)$$

Rewriting this last term, and factoring out a factor of $\frac{1}{i^2}$:

$$\langle 0|TQ(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle = -\frac{1}{i^2}\left(\frac{1}{2m\omega}\right)^2\left(e^{i\omega(t_4-t_1)}e^{i\omega(t_2-t_3)} + e^{i\omega(t_4-t_1)}e^{i\omega(t_3-t_2)}\right.$$

$$+e^{i\omega(t_4-t_2)}e^{i\omega(t_3-t_1)}\big)$$

which gives:

$$\langle 0|TQ(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle = \frac{1}{i^2}\left(\frac{1}{2m\omega}\right)^2 \big(ie^{-i\omega(t_1-t_2)}ie^{-i\omega(t_3-t_4)} + ie^{-i\omega(t_1-t_4)}ie^{-i\omega(t_2-t_3)}$$

$$+ie^{-i\omega(t_2-t_4)}ie^{-i\omega(t_1-t_3)}\big)$$

As before, we've assumed that the time ordering was as written. If the time ordering is backward, we can flip the offending terms, and the exponential will go positive. As a result, we can put in absolute value signs to account for all possible orderings:

$$\langle 0|TQ(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle = \frac{1}{i^2}\left(\frac{1}{2m\omega}\right)^2 \big(ie^{-i\omega|t_1-t_2|}ie^{-i\omega|t_3-t_4|} + ie^{-i\omega|t_1-t_4|}ie^{-i\omega|t_2-t_3|}$$

$$+ie^{-i\omega|t_2-t_4|}ie^{-i\omega|t_1-t_3|}\big)$$

which gives:

$$\langle 0|TQ(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle = \frac{1}{i^2}\left[G(t_1-t_2)G(t_3-t_4) + G(t_1-t_4)G(t_2-t_3)\right.$$

$$\left.+G(t_2-t_4)G(t_1-t_3)\right]$$

which is equation 7.17. □

Srednicki 7.4. Consider a harmonic oscillator in its ground state at $\mathbf{t = -\infty}$. It is then subjected to an external force $\mathbf{f(t)}$. Compute the probability $|\langle \mathbf{0|0}\rangle_\mathbf{f}|^\mathbf{2}$ that the oscillator is still in its ground state at $\mathbf{t = \infty}$. Write your answer as a manifestly real expression, and in terms of the Fourier transform $\widetilde{\mathbf{f}}(\mathbf{E}) = \int_{-\infty}^{\infty} \mathbf{dte^{iEt}f(t)}$. Your answer should not involve any other unevaluated integrals.

The answer is Srednicki 7.10:

$$\langle 0|0\rangle_f = exp\left[\frac{i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{\widetilde{f}(E)\widetilde{f}(-E)}{-E^2+\omega^2-i\varepsilon}\right]$$

The remainder of the problem is just to simplify this and put it into the form requested. First we note that

$$\widetilde{f}(E) = \int_{-\infty}^{\infty} dte^{iEt}f(t)$$

so

$$\widetilde{f}(-E) = \int_{-\infty}^{\infty} dte^{-iEt}f(t) \tag{4.3.1}$$

and

$$\widetilde{f}^*(E) = \int_{-\infty}^{\infty} dte^{-iEt}f(t) \tag{4.3.2}$$

Hence $\widetilde{f}(-E) = \widetilde{f}^*(E)$. Then,

$$\langle 0|0\rangle_f = exp\left[\frac{i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{\widetilde{f}(E)\widetilde{f}^*(E)}{-E^2+\omega^2-i\varepsilon}\right]$$

which is:

$$\langle 0|0\rangle_f = exp\left[\frac{i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{|\widetilde{f}(E)|^2}{-E^2+\omega^2-i\varepsilon}\right]$$

This is of the form: $e^{i(a+ib)}$. Now that we take the magnitude of this, we have $e^{i(a+ib)}e^{-i(a-ib)} = e^{-2b}$. Hence,

$$|\langle 0|0\rangle_f|^2 = exp\left[-Im\left\{\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{|\widetilde{f}(E)|^2}{-E^2+\omega^2-i\varepsilon}\right\}\right]$$

Most of this is already manifestly real, so let's move that outside of the Imaginary Operator. Then,

$$|\langle 0|0\rangle_f|^2 = exp\left[-\int_{-\infty}^{\infty}\frac{dE}{2\pi}|\widetilde{f}(E)|^2\ Im\left\{\frac{1}{-E^2+\omega^2-i\varepsilon}\right\}\right]$$

This is:

$$|\langle 0|0\rangle_f|^2 = exp\left[-\int_{-\infty}^{\infty}\frac{dE}{2\pi}|\widetilde{f}(E)|^2\ Im\left\{\frac{1}{-E^2+\omega^2-i\varepsilon}\cdot\frac{-E^2+\omega^2+i\varepsilon}{-E^2+\omega^2+i\varepsilon}\right\}\right]$$

$$|\langle 0|0\rangle_f|^2 = exp\left[-\int_{-\infty}^{\infty}\frac{dE}{2\pi}|\widetilde{f}(E)|^2\ Im\left\{\frac{-E^2+\omega^2+i\varepsilon}{(-E^2+\omega^2)^2+\varepsilon^2}\right\}\right]$$

Taking the imaginary component:

$$|\langle 0|0\rangle_f|^2 = exp\left[-\int_{-\infty}^{\infty}\frac{dE}{2\pi}|\widetilde{f}(E)|^2\ \frac{\varepsilon}{(-E^2+\omega^2)^2+\varepsilon^2}\right]$$

Let's define $a = \omega^2 - E^2$. Then:

$$|\langle 0|0\rangle_f|^2 = exp\left[-\int_{-\infty}^{\infty}\frac{dE}{2\pi}|\widetilde{f}(E)|^2\ \frac{\varepsilon}{a^2+\varepsilon^2}\right]$$

Now we're ready to let $\varepsilon \to 0$. We notice that:

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{a^2+\varepsilon^2} = \begin{cases} 0 \text{ if } a \neq 0 \\ \infty \text{ if } a = 0 \end{cases}$$

This is a delta function by definition. But any delta function will work, so:

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{a^2+\varepsilon^2} = C\delta(a)$$

If we integrate both sides from $-\infty$ to ∞, we find $\pi =$ C. Then,

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{a^2+\varepsilon^2} = \pi\delta(a)$$

Hence:

$$|\langle 0|0\rangle_f|^2 = exp\left[-\int_{-\infty}^{\infty}\frac{dE}{2\pi}|\widetilde{f}(E)|^2\pi\delta(E^2-\omega^2)\right]$$

Using a property of the delta function:

$$|\langle 0|0\rangle_f|^2 = exp\left[-\int_{-\infty}^{\infty}\frac{dE}{2}|\widetilde{f}(E)|^2\left(\frac{\delta(E-\omega)}{|2E|}+\frac{\delta(E+\omega)}{|2E|}\right)\right]$$

This is:

$$|\langle 0|0\rangle_f|^2 = exp\left[-\int_{-\infty}^{\infty}\frac{dE}{2}|\widetilde{f}(E)|^2\frac{\delta(E-\omega)}{2|E|}\right] exp\left[-\int_{-\infty}^{\infty}\frac{dE}{2}|\widetilde{f}(E)|^2\frac{\delta(E+\omega)}{2|E|}\right]$$

Performing the integrals:

$$|\langle 0|0\rangle_f|^2 = exp\left[-\frac{1}{2}|\widetilde{f}(\omega)|^2\frac{1}{2\omega}\right] exp\left[-\frac{1}{2}|\widetilde{f}(-\omega)|^2\frac{1}{2\omega}\right]$$

Now we noticed previously (equations (4.3.1) and (4.3.2)) that $\widetilde{f}(-x) = \widetilde{f}^*(x)$. Consequently, $|\widetilde{f}(-x)|^2 = |\widetilde{f}^*(x)|^2 = |\widetilde{f}(x)|^2$, which gives:

$$|\langle 0|0\rangle_f|^2 = exp\left[-\frac{1}{2}|\widetilde{f}(\omega)|^2\frac{1}{2\omega}\right] exp\left[-\frac{1}{2}|\widetilde{f}(\omega)|^2\frac{1}{2\omega}\right]$$

$$|\langle 0|0\rangle_f|^2 = exp\left[-\frac{|\widetilde{f}(\omega)|^2}{2\omega}\right]$$