Srednicki Chapter 68

QFT Problems & Solutions

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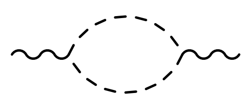
Srednicki 68.1. Consider the current correlation function $\langle 0|Tj^\mu(x)j^\nu(y)|0\rangle$ in spinor electrodynamics.

(a) Show that its Fourier transform is proportional to

$$\Pi^{\mu\nu}(k) + \Pi^{\mu\rho}(k)\widetilde{\Delta}_{\rho\sigma}(k)\Pi^{\sigma\nu}(k) + \ldots$$

Recall from the second paragraph that $Z_1 j^\mu$ inside a correlation function is a vertex to which an external photon can connect.

So, two such vertices will give the following diagram:



where the dashed circles represent a "black box" that can be anything (though this has to be a vertex is defined in rule 7 in section 58; ie there have to be fermions involved). More specifically, this can be a 1PI diagram, or several 1PI diagrams connected by a photon. We therefore have:

$$Z_1^2\langle 0|Tj^\mu j^\nu|0\rangle = \Pi^{\mu\nu}(k) + \Pi^{\mu\rho}(k)\widetilde{\Delta}_{\rho\sigma}\Pi^{\sigma\nu}(k) + \ldots$$

which shows the proportionality we expected.

(b) Use this to prove that $\Pi^{\mu\nu}(k)$ is transverse: $k_\mu\Pi^{\mu\nu} = 0$.

Recall from the previous section that:

$$k_\mu\mathcal{M}^\mu = 0$$

Now remember that $\mathcal{M}$ is what we get from the correlation functions, which we evaluate through the Feynman Diagrams, which are drawn in momentum space. So we take the Fourier Transform to write this in position space:

$$\partial_\mu\langle 0|Tj^\mu(x)j^\nu(x)|0\rangle = 0$$

Now we take the Fourier Transform again to get back to momentum space, and we have from the result in part (a) that:

$$k_\mu \left(\Pi^{\mu\nu} + \Pi^{\mu\rho}(k)\widetilde{\Delta}_{\rho\sigma}\Pi^{\sigma\nu}(k) + \dots\right)$$

Nothing in general will cancel with this first term, so it must go to zero on its own. Thus:

$$k_\mu \Pi^{\mu\nu} = 0$$

ie that Π is transverse, as expected.

Srednicki 68.2. Verify that equation 68.12 holds at the one-loop level with $Z_1 = Z_2$.

The exact vertex function is given by:

$$V^\mu(p', p) = eZ_1\gamma^\mu + V_{1-\text{loop}}(p', p) + \dots$$

which we get just from the Lagrangian in 62.2. Then:

$$(p' - p)_\mu V^\mu(p', p) = eZ_1(\not{p}' - \not{p}) + (p' - p)_\mu V^\mu_{1-\text{loop}}(p', p) \tag{68.2.1}$$

which at least gives us an equation we can work with. To make progress, we need an equation for the one-loop term – and we need it to be gauge-independent! Let's use 62.40 (we insert the missing factor of Z_1^3:

$$V^\mu_{1-\text{loop}}(p', p) = -i(Z_1 e)^3 \int \frac{d^4\ell}{(2\pi)^4}\left[\gamma^\rho \widetilde{S}(\not{p}' + \not{\ell})\gamma^\mu \widetilde{S}(\not{p} + \not{\ell})\gamma^\nu\right]\widetilde{\Delta}_{\nu\rho}(\ell)$$

This gives:

$$(p' - p)_\mu V^\mu_{1-\text{loop}}(p', p) = -i(Z_1 e)^3 \int \frac{d^4\ell}{(2\pi)^4}\left[\gamma^\rho \widetilde{S}(\not{p}' + \not{\ell})(\not{p}' - \not{p})\widetilde{S}(\not{p} + \not{\ell})\gamma^\nu\right]\widetilde{\Delta}_{\nu\rho}(\ell) \tag{68.2.2}$$

Just for fun, let's rewrite this as:

$$(p'-p)_\mu V^\mu_{1-\text{loop}}(p', p) = -i(Z_1 e)^3 \int \frac{d^4\ell}{(2\pi)^4}\left[\gamma^\rho \widetilde{S}(\not{p}' + \not{\ell})(\not{p}' + \ell + m - \not{p} - \ell - m)\widetilde{S}(\not{p} + \not{\ell})\gamma^\nu\right]\widetilde{\Delta}_{\nu\rho}(\ell) \tag{68.2.3}$$

Now we recall $\widetilde{S}$ represents an internal fermion written in momentum space. By the Feynman Rules, the value is:

$$\widetilde{S}(p) = \frac{-\not{p} + m}{p^2 + m^2 - i\varepsilon}$$

Note that we dropped the $-i$, since the feynman rules give the value of the diagram multiplied by i. We drop the infinitesimal part, implicitely promising not to allow the denominator to vanish. We then use 47.10 to rewrite this as:

$$\widetilde{S}(p) = \frac{-\not{p} + m}{(-\not{p} + m)(\not{p} + m)}$$

and so:

$$\widetilde{S}(p) = \frac{1}{\not{p} + m}$$

Now we use this in (68.2.3):

$$(p'-p)_\mu V^\mu_{1-\mathrm{loop}}(p', p) = -i(Z_1 e)^3 \int \frac{d^4\ell}{(2\pi)^4} \left[\gamma^\rho \widetilde{S}(\not{p}' + \not{\ell})(\widetilde{S}(\not{p}' + \ell)^{-1} - \widetilde{S}(\not{p} + \ell)^{-1})\widetilde{S}(\not{p} + \not{\ell})\gamma^\nu\right] \widetilde{\Delta}_{\nu\rho}(\ell)$$

Cancelling some stuff:

$$(p' - p)_\mu V^\mu_{1-\mathrm{loop}}(p', p) = -i(Z_1 e)^3 \int \frac{d^4\ell}{(2\pi)^4} \gamma^\rho \left(\widetilde{S}(\not{p} + \not{\ell}) - \widetilde{S}(\not{p}' + \ell)\right) \gamma^\nu \widetilde{\Delta}_{\nu\rho}(\ell)$$

Now we use equation 62.28:

$$(p' - p)_\mu V^\mu_{1-\mathrm{loop}}(p', p) = Z_1 e \left[\Sigma(\not{p}) - \Sigma(\not{p}') + (Z_2 - 1)(\not{p} - \not{p}') + \ldots\right]$$

Notice that each term here is at least of order e^3 already. To avoid going to higher order, we approximate $Z_1 = 1$, absorbing higher terms into the ellipses. Then:

$$(p' - p)_\mu V^\mu_{1-\mathrm{loop}}(p', p) = e \left[\Sigma(\not{p}) - \Sigma(\not{p}') + (Z_2 - 1)(\not{p} - \not{p}') + \ldots\right]$$

Now we're ready to put this back into equation (68.2.1):

$$(p' - p)_\mu V^\mu(p', p) = eZ_1(\not{p}' - \not{p}) + e \left[\Sigma(\not{p}) - \Sigma(\not{p}') + (Z_2 - 1)(\not{p} - \not{p}') + \ldots\right]$$

If $Z_1 = Z_2$, then:

$$(p' - p)_\mu V^\mu(p', p) = e \left[\Sigma(\not{p}) - \Sigma(\not{p}') - (\not{p} - \not{p}') + \ldots\right]$$

which is:

$$(p' - p)_\mu V^\mu(p', p) = e \left[\Sigma(\not{p}) - \not{p} - \Sigma(\not{p}') + \not{p}' + \ldots\right]$$

Thus:

$$(p' - p)_\mu V^\mu(p', p) = e \left[-\left(\not{p} + m - \Sigma(\not{p})\right) + (\not{p}' + m - \Sigma(\not{p}')) + \ldots\right]$$

Finally we use 62.27, ignoring the infinitesimal (or adding and subtracting it, as you like):

$$(p' - p)_\mu V^\mu(p', p) = e\widetilde{\boldsymbol{S}}(\not{p}')^{-1} - e\widetilde{\boldsymbol{S}}(p)^{-1}$$

which is 68.12, given that $Z_1 = Z_2$.

Note: Be careful not to confuse the exact fermion propagator $\boldsymbol{S}$ with the internal tree-level propagator S. The symbols are the same up to the boldface.

Note 2: This problem is extremely challenging without further hints. All the steps are easy enough, but writing this argument was not at all obvious: we had to use results across several chapters, including the simplification of the internal propagator, which was never done in the text.

Note 3: There is a slight mistake in Srednicki's equation 68.20 in his solutions; the correction is obvious in my solution.

Srednicki 68.3. Scalar Electrodynamics.

(a) Consider the Fourier Transform of $\langle 0|TJ^\mu(x)\phi(y)\phi^\dagger(z)|0\rangle$, where:

$$\boldsymbol{J^\mu = -ieZ_2[\phi^\dagger\partial^\mu\phi - (\partial^\mu\phi^\dagger)\phi] - 2Z_1e^2A^\mu\phi^\dagger\phi}$$

is the Noether current. You may assume that $Z_4 = Z_1^2/Z_2$ (which is necessary for gauge invariance). Show that:

$$\boldsymbol{(p'-p)_\mu V_3^\mu(p',p) = Z_2^{-1}Z_1e\left[\widetilde{\Delta}(p')^{-1} - \widetilde{\Delta}(p)^{-1}\right]}$$

where $V_3^\mu(p',p)$ is the exact scalar-scalar-photon vertex function, and $\widetilde{\Delta}(p)$ is the exact scalar propagator.

As instructed, we consider the Fourier Transform of this, getting something analogous to 68.1:

$$C^\mu(k,p',p) = \int d^4xd^4yd^4ze^{ik_1x-ik_2y+ik_3z}\langle 0|TJ^\mu(x)\phi(y)\phi^\dagger(z)|0\rangle$$

Now we plug in the conserved current:

$$C^\mu(k,p',p) = \int d^4xd^4yd^4ze^{ik_1x-ik_2y+ik_3z}\left\{(-ieZ_2)\langle 0|T\phi^\dagger(x)\partial^\mu\phi(x)\phi(y)\phi^\dagger(z)|0\rangle - \right.$$

$$\left.(-ieZ_2)\langle 0|T\partial^\mu\phi^\dagger(x)\phi(x)\phi(y)\phi^\dagger(z)|0\rangle + (-2Z_1e^2A^\mu)\langle 0|T\phi^\dagger(x)\phi(x)\phi(y)\phi^\dagger(z)|0\rangle\right\}$$

Now we consider:

$$\phi(x) = \int d^4ke^{-ikx}\widetilde{\phi}(k)$$

$$\partial^\mu\phi(x) = \partial^\mu\int d^4ke^{-ikx}\widetilde{\phi}(k)$$

$$\partial^\mu\phi(x) = -i\int d^4ke^{-ikx}\widetilde{\phi}(k)$$

Now we need to be very careful. In principle, this is the end of the line. However, in our equation, we are Fourier Transforming over the xs, so any function of x is really a function of k. Therefore, there is no harm in writing this as

$$\partial^\mu\phi(x) = -ik\phi(x)$$

Though this is NOT true in general.

Plugging this in, we have:

$$C^\mu(k,p',p) = \int d^4xd^4yd^4ze^{ik_1x-ik_2y+ik_3z}\left\{(-ek^\mu Z_2)\langle 0|T\phi^\dagger(x)\phi(x)\phi(y)\phi^\dagger(z)|0\rangle - \right.$$

$$(k^\mu e Z_2)\langle 0|T\phi^\dagger(x)\phi(x)\phi(y)\phi^\dagger(z)|0\rangle + (-2Z_1 e^2 A^\mu)\langle 0|T\phi^\dagger(x)\phi(x)\phi(y)\phi^\dagger(z)|0\rangle\}$$

These terms now combine:

$$C^\mu(k,p',p) = -2\int d^4x d^4y d^4z e^{ik_1x-ik_2y+ik_3z}(ek^\mu Z_2 + e^2 A^\mu Z_1)\langle 0|T\phi^\dagger(x)\phi(x)\phi(y)\phi^\dagger(z)|0\rangle$$

Now we use the solution to problem 8.8 to get the correlation functions:

$$C^\mu(k,p',p) = 2\int d^4x d^4y d^4z e^{ik_1x-ik_2y+ik_3z}(ek^\mu Z_2 + e^2 A^\mu Z_1)\Delta(x-z)\Delta(y-x)$$

Now we write the exponential slightly differently:

$$C^\mu(k,p',p) = 2\int d^4x d^4y d^4z e^{i[-k_3(x-z)-k_2(y-x)+(k_1-k_2+k_3)x]}(ek^\mu Z_2 + e^2 A^\mu Z_1)\Delta(x-z)\Delta(y-x)$$

Now I want to define $a = \frac{1}{\sqrt{2}}(x-z)$, $b = \frac{1}{\sqrt{2}}(y-z)$, and $c = x$ in order to proceed. However, this is a non-orthogonal basis, and so it is not possible to do these integrals as we would need to in order to get the desired answer.

So how does Srednicki get his answer? He bypasses much of this math by using physical intuition, and while I believe that his work is correct, I also believe that his technique is not fully general. I could be wrong, but I think that he got lucky in that passing over these difficulties did not affect the outcome. Therefore, I will not reproduce his solution here.

Please feel free to e-mail with thoughts on this problem.