Srednicki 66.1. Compute the one-loop contributions to the anomalous dimensions of $m$, $\Psi$, and $A^\mu$ in spinor electrodynamics in Feynman gauge.

We begin with this calculation, which we will need later. By equation 66.23:

$$\ln e_0 = \sum_{n=1}^\infty \frac{E_n(e, \lambda)}{\varepsilon^n} + \ln e + \varepsilon \ln \tilde{\mu}$$

We take the derivative of both sides with respect to $\ln \mu$. The left hand side is a bare parameter which should not depend on $\mu$, so that vanishes. The sum also vanishes, because a renormalizable theory should be well-defined around $\varepsilon = 0$. The other two terms can then be equated:

$$\frac{1}{e} \frac{de}{\ln \mu} = -\frac{\varepsilon}{2} \frac{d \ln \tilde{\mu}}{d \ln \mu}$$

$\mu$ and $\tilde{\mu}$ are different only by a constant, so this derivative will vanish. Then:

$$\frac{de}{\ln \mu} = -\frac{e\varepsilon}{2} \quad (66.1.1)$$

Now we have, by definition:

$$\gamma_m = \frac{d \ln m}{d \ln \mu}$$

Notice that $\mathcal{L} \sim Z_2 \bar{\Psi}\Psi$. This implies that $\Psi_0 = Z_2^{1/2}\Psi$. Further, we have: $\mathcal{L} \sim Z_m m \bar{\Psi}\Psi$, which gives us that $m_0 = mZ_m/Z_2$. Then:

$$\gamma_m = \frac{d}{d \ln \mu} [\ln m_0 + \ln (Z_2/Z_m)]$$

This bare field does not depend on $\mu$, so we can drop this term:

$$\gamma_m = \frac{d}{d \ln \mu} \ln (Z_2/Z_m)$$

Using the chain rule:

$$Z_m = \frac{d}{d \ln \mu} (Z_2/Z_m) \frac{de}{d \ln \mu}$$
Now we use equation (66.1.1):

\[ Z_m = -\frac{e\varepsilon}{2} \frac{d}{d\varepsilon} (\ln Z_2 - \ln Z_m) \]

Now we use 62.34 and 62.35, and recall \( \ln(1 + x) = x + \ldots \) Then:

\[ Z_m = -\frac{e\varepsilon}{2} \frac{d}{d\varepsilon} \left( -\frac{e^2}{8\pi^2\varepsilon} + \frac{e^2}{2\pi^2\varepsilon} \right) \]

which is:

\[ Z_m = -\frac{e\varepsilon}{2} \frac{d}{d\varepsilon} \left( \frac{3e^2}{8\pi^2\varepsilon} \right) \]

Taking the derivative:

\[ Z_m = -\frac{3e^2}{8\pi^2} \]

Now for the fields; we have:

\[ \gamma_\Psi = \frac{1}{2} \frac{d\ln Z_\Psi}{d\ln \mu} \]

In this case, \( Z_\Psi = Z_2 \), where \( Z_2 = 1 - \frac{e^2}{8\pi^2\varepsilon} \). This depends only on \( e \); therefore we use the chain rule:

\[ \gamma_\Psi = \frac{1}{2} \frac{d\ln Z_2}{d\varepsilon} \frac{d\varepsilon}{d\ln \mu} \]

Using equation (66.1.1)

\[ \gamma_\Psi = -\frac{e\varepsilon}{4} \frac{d\ln Z_2}{d\varepsilon} \]

Expanding:

\[ \gamma_\Psi = -\frac{e\varepsilon}{4} \frac{d}{d\varepsilon} \left[ -\frac{e^2}{8\pi^2\varepsilon} \right] \]

Taking the derivative:

\[ \gamma_\Psi = -\frac{e^2}{16\pi^2} \]

As for \( A \), we proceed as before:

\[ \gamma_A = -\frac{e\varepsilon}{4} \frac{d\ln Z_3}{d\varepsilon} \]

Using 62.24, expanding, and setting aside the finite portion (this will vanish anyway when \( \varepsilon \to 0 \)):

\[ \gamma_A = -\frac{e\varepsilon}{4} \frac{-e}{3\pi^2\varepsilon} \]

which is:

\[ \gamma_A = -\frac{e^2}{12\pi^2} \]

Srednicki 66.2. Compute the one-loop contributions to the anomalous dimensions of \( m, \phi, \) and \( A^\mu \) in scalar electrodynamics.
As before, we relate the bare fields to the renormalized fields to determine that \( \phi_0 = Z_2^{1/2} \phi \), and further that \( m_0 = Z_m^{1/2} Z_2^{-1/2} m \). This gives \( \ln m_0 = \ln m + \tfrac{1}{2} \ln \left( \frac{Z_m}{Z_2} \right) \). Then,

\[
\gamma_{m} = \frac{d \ln m}{d \ln \mu}
\]

which is:

\[
\gamma_{m} = \frac{d}{d \ln \mu} \left[ \ln m_0 - \frac{1}{2} \ln \left( \frac{Z_m}{Z_2} \right) \right]
\]

The bare parameters should be independent of \( \mu \), and so:

\[
\gamma_{m} = -\frac{1}{2} \frac{d}{d \ln \mu} \ln(Z_m Z_2^{-1})
\]

Expanding the logarithm about 0:

\[
\gamma_{m} = -\frac{1}{2} \frac{d}{d \ln \mu} \left[ \frac{\lambda}{8\pi^2 \varepsilon} - \frac{3\epsilon^2}{8\pi^2 \varepsilon} \right]
\]

Using the chain rule, we have:

\[
\gamma_{m} = -\frac{1}{2} \left[ \frac{d}{de} \frac{d}{d \ln \mu} + \frac{d}{d\lambda} \frac{d}{d \ln \mu} \right] \left[ \frac{\lambda}{8\pi^2 \varepsilon} - \frac{3\epsilon^2}{8\pi^2 \varepsilon} \right]
\]

Equation (66.1.1) gives \( \frac{de}{d \ln \mu} = -\frac{\varepsilon}{2} \), and performing the analogous operation to equation 66.24 gives \( \frac{d\lambda}{d \ln \mu} = -\lambda \varepsilon \). This gives:

\[
-\frac{1}{2} \left[ -\frac{\varepsilon}{2} \frac{d}{de} - \lambda \varepsilon \frac{d}{d\lambda} \right] \left[ \frac{\lambda}{8\pi^2 \varepsilon} - \frac{3\epsilon^2}{8\pi^2 \varepsilon} \right]
\]

Doing the derivatives and simplifying, we have:

\[
\begin{align*}
\gamma_{m} &= \frac{\lambda - 3\epsilon^2}{16\pi^2} \\
\gamma_{\phi} &= \frac{1}{2} \frac{d \ln Z_\phi}{d \ln \mu}
\end{align*}
\]

Similarly, we have:

\[
\gamma_{\phi} = \frac{1}{2} \frac{d \ln Z_\phi}{d \ln \mu}
\]

Equation 65.25 tells us that \( Z_\phi = Z_2 \) in this case. Thus,

\[
\gamma_{\phi} = \frac{1}{2} \frac{d \ln Z_2}{d \ln \mu}
\]

Using the chain rule:

\[
\gamma_{\phi} = \frac{1}{2} \frac{d \ln Z_2}{d \ln \mu} \frac{d e}{d \ln \mu}
\]
Solving this, using equation (66.1.1):

\[
\gamma_\phi = \frac{1}{2} \left[ \frac{3e}{4\pi^2\varepsilon} \right] \left[ -\frac{e\varepsilon}{2} \right]
\]

Simplifying this, we have:

\[
\gamma_\phi = -\frac{3e^2}{16\pi^2}
\]

Now we repeat this for \(A\), which has \(Z_3\) as a coefficient. We have:

\[
\gamma_A = \frac{1}{2} \frac{d\ln Z_3}{de} \frac{de}{d\ln \mu}
\]

This gives:

\[
\gamma_A = \frac{e^2}{48\pi^2}
\]

Srednicki 66.3. Use the results of problem 62.2 to compute the anomalous dimensions of \(m\) and the beta function for \(e\) in spinor electrodynamics in \(R_\xi\) gauge. You should find that the results are independent of \(\xi\).

From 62.3, we have \(\Psi_0 = Z_2^{1/2}\Psi\), and so \(m_0 = Z_2^{-1}Z_m m\). Taking the logarithm, we have

\[
\ln m_0 = \ln m + \ln Z_m Z_2^{-1}
\]

Taking the derivative, we have:

\[
\frac{d\ln m_0}{d\ln \mu} = \frac{d\ln m}{d\ln \mu} + \frac{d\ln(Z_m/Z_2)}{d\ln \mu}
\]

The bare fields are independent of \(\mu\), so:

\[
\frac{d\ln m}{d\ln \mu} = -\frac{d}{d\ln \mu} \ln [Z_m/Z_2]
\]

Now \(\gamma_m = \frac{d\ln m}{d\ln \mu}\), so

\[
\gamma_m = -\frac{d}{d\ln \mu} [\ln Z_m - \ln Z_2]
\]

Using the chain rule and our results from problem 62.2, we have:

\[
\gamma_m = -\frac{d}{de} \frac{d\ln m}{d\ln \mu} \left[ \left(1 - \frac{e^2(3 + \xi)}{8\pi^2\varepsilon}\right) - \ln \left(1 - \frac{e^2\xi}{8\pi^2\varepsilon}\right) \right]
\]

From equation (66.1.1) (the derivation of which stands on its own), we have \(\frac{de}{d\ln \mu} = -\frac{e\varepsilon}{2}\). Also recall that \(\ln(1 + x) = x + \ldots\). Thus:

\[
\gamma_m = -\left(-\frac{e\varepsilon}{2}\right) \frac{d}{d\varepsilon} \left[ -\frac{e^2(3 + \xi)}{8\pi^2\varepsilon} + \frac{e^2\xi}{8\pi^2\varepsilon} \right]
\]
This gives:
\[ \gamma_m = \frac{e\varepsilon}{2} \left[ -\frac{3e}{4\pi^2\varepsilon} \right] \]
which is:
\[ \gamma_m = \left[ -\frac{3e^2}{8\pi^2} \right] \]
Now for the beta function. Matching bare fields with renormalized fields, we have \( \Psi_0 = Z_2^{1/2} \Psi \), which implies \( m_0 = Z_2^{-1}Z_mm \). Further, \( A_0 = Z_3^{1/2} A \). These imply that:
\[ e_0 = Z_2^{-1}Z_3^{-1/2}Z_1 e \]
Now we should shift the mass dimensionality off of \( e \). We have \( \mathcal{L} \sim \overline{\Psi} \phi \Psi \); the \( \partial \) has a mass dimensionality of 1, so the \( \Psi \) must have mass dimensionality of \((d-1)/2\). The last term in equation 62.1 reduces to \( \mathcal{L} \sim \partial^\mu A^\nu \partial_\mu A_\nu \), so \( A \) has mass dimensionality of \((d-2)/2\). Now consider the term \( \mathcal{L} \sim e \overline{\Psi} A \Psi \). Using this, we see that in four dimensions, \( [e] = 0 \), and in six dimensions, \( [e] = -1 \). Thus, \( [e] = \varepsilon/2 \), where \( \varepsilon = 4 - d \). Thus:
\[ e_0 = Z_2^{-1}Z_3^{-1/2}Z_1 e^{\varepsilon/2} \]
Using the result from problem 62.2, we have:
\[ \ln Z_1 = -\frac{e^2\xi}{8\pi^2\varepsilon} + \ldots \]
\[ \ln Z_2^{-1} = \frac{e^2\xi}{8\pi^2\varepsilon} + \ldots \]
\[ \ln Z_3^{-1/2} = \frac{e^2}{12\pi^2\varepsilon} + \ldots \]
This gives:
\[ \ln Z_1 Z_2^{-1} Z_3^{-1/2} = E = \sum_{i=1}^{\infty} \frac{E_i}{\varepsilon^i} = \frac{e^2}{12\pi^2\varepsilon} + \ldots \]
Now we have:
\[ \ln e_0 = E + \ln e + \frac{\varepsilon}{2} \ln \mu \]
Taking the derivative with respect to \( \ln \mu \), and using the chain rule:
\[ 0 = \frac{\partial E}{\partial e} \frac{\partial e}{\partial \ln \mu} + \frac{\partial \ln e}{\partial \ln \mu} + \varepsilon/2 \]
Take the derivative in the second term:
\[ 0 = \frac{\partial E}{\partial e} \frac{\partial e}{\partial \ln \mu} + \frac{\partial e}{\partial \ln \mu} + \varepsilon/2 \]
Now we factor:
\[ 0 = \left( e \frac{\partial E}{\partial e} + 1 \right) \frac{\partial e}{\partial \ln \mu} + \frac{e\varepsilon}{2} \]
Now we use our usual “physical reasoning” trick: a renormalizable theory should be well defined as $\varepsilon \to 0$, so we can set aside the $E$ term. Then:

$$\frac{de}{d\ln \mu} = -\frac{e\varepsilon}{2} + \beta(e)$$

This gives:

$$0 = \left( e \frac{\partial E}{\partial e} + 1 \right) \left( -\frac{e\varepsilon}{2} + \beta(e) \right) + \frac{e\varepsilon}{2}$$

Now we distribute, and match up the terms with no $\varepsilon$s. We find:

$$e \frac{\partial E_1}{\partial e} \varepsilon \left( -\frac{e\varepsilon}{2} \right) + \beta(e) = 0$$

This gives:

$$\left[ \beta(e) = \frac{e^3}{12\pi^2} \right]$$

Srednicki 66.4. The value of $\alpha(M_W)$. The solution of equation 66.12 is:

$$\frac{1}{\alpha(M_W)} = \frac{1}{\alpha(\mu)} - \frac{2}{3\pi} \sum_i Q_i^2 \ln \left( \frac{M_W}{\mu} \right)$$

where the sum is over all quarks and leptons (each color of quark counts separately), and we have chosen the $W^{\pm}$ boson mass $M_W$ as a reference scale. We can define a different renormalization scheme, modified decoupling subtraction or DS, where we imagine integrating out a field when $\mu$ is below its mass. In this scheme, equation 66.30 becomes:

$$\frac{1}{\alpha(M_W)} = \frac{1}{\alpha(\mu)} - \frac{2}{3\pi} \sum_i Q_i^2 \ln [M_W/\min(m_i, \mu)]$$

where the sum is now over all quarks and leptons with mass less that $M_W$. For $\mu < m_e$, the DS, scheme coincides with the OS scheme, and we have

$$\frac{1}{\alpha(M_W)} = \frac{1}{\alpha} - \frac{2}{3\pi} \sum_i Q_i^2 \ln (M_W/m_i)$$

where $\alpha = 1/137.036$ is the fine-structure constant in the OS scheme. Using $m_\mu = m_d = m_s \sim 300$ MeV for the light quark masses (because quarks should be replaced by hadrons at lower energies), and other quark and lepton masses from sections 83 and 88, compute $\alpha(M_W)$.

It’s just a matter of plugging into the formula. We have:

$$\frac{1}{\alpha(M_W)} = 137.036 - \frac{2}{3\pi} \left\{ (\pm1)^2 \ln \left[ \frac{M_W^3}{0.511 \cdot 105.7 \cdot 1777} \right] + 3 \cdot \left( \frac{2}{3} \right)^2 \ln \left[ \frac{M_W^3}{300 \cdot 1300 \cdot 174000} \right] + \ldots \right\}$$
\[ 3 \cdot \left(\frac{1}{3}\right)^2 \ln \left[ \frac{M_w^3}{300 \cdot 300 \cdot 4300} \right] \]

which is:

\[ \frac{1}{\alpha(M_w)} = 128.7 \]