Srednicki Chapter 66

QFT Problems & Solutions

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Srednicki 66.1. Compute the one-loop contributions to the anomalous dimensions of m, Ψ, and A^μ in spinor electrodynamics in Feynman gauge.

We begin with this calculation, which we will need later. By equation 66.23:

$$\ln e_0 = \sum_{n=1}^{\infty} \frac{E_n(e,\lambda)}{\varepsilon^n} + \ln e + \frac{\varepsilon}{2}\ln\tilde{\mu}$$

We take the derivative of both sides with respect to $\ln\mu$. The left hand side is a bare parameter which should not depend on μ, so that vanishes. The sum also vanishes, because a renormalizable theory should be well-defined around $\varepsilon = 0$. The other two terms can then be equated:

$$\frac{1}{e}\frac{de}{\ln\mu} = -\frac{\varepsilon}{2}\frac{d\ln\tilde{\mu}}{d\ln\mu}$$

μ and $\tilde{\mu}$ are different only by a constant, so this derivative will vanish. Then:

$$\frac{de}{\ln\mu} = -\frac{e\varepsilon}{2} \tag{66.1.1}$$

Now we have, by definition:

$$\gamma_m = \frac{d\ln m}{d\ln\mu}$$

Notice that $\mathcal{L} \sim Z_2\overline{\Psi}\partial\!\!\!/\Psi$. This implies that $\Psi_0 = Z_2^{1/2}\Psi$. Further, we have: $\mathcal{L} \sim Z_m m\overline{\Psi}\Psi$, which gives us that $m_0 = mZ_m/Z_2$. Then:

$$\gamma_m = \frac{d}{d\ln\mu}\left[\ln m_0 + \ln\left(Z_2/Z_m\right)\right]$$

This bare field does not depend on μ, so we can drop this term:

$$\gamma_m = \frac{d}{d\ln\mu}\ln\left(Z_2/Z_m\right)$$

Using the chain rule:

$$Z_m = \frac{d\ln(Z_2/Z_m)}{de}\frac{de}{d\ln\mu}$$

Now we use equation (66.1.1):

$$Z_m = -\frac{e\varepsilon}{2}\frac{d}{de}(\ln Z_2 - \ln Z_m)$$

Now we use 62.34 and 62.35, and recall $\ln(1+x) = x + \dots$. Then:

$$Z_m = -\frac{e\varepsilon}{2}\frac{d}{de}\left(-\frac{e^2}{8\pi^2\varepsilon} + \frac{e^2}{2\pi^2\varepsilon}\right)$$

which is:

$$Z_m = -\frac{e\varepsilon}{2}\frac{d}{de}\left(\frac{3e^2}{8\pi^2\varepsilon}\right)$$

Taking the derivative:

$$\boxed{Z_m = -\frac{3e^2}{8\pi^2}}$$

Now for the fields; we have:

$$\gamma_\Psi = \frac{1}{2}\frac{d\ln Z_\Psi}{d\ln\mu}$$

In this case, $Z_\Psi = Z_2$, where $Z_2 = 1 - \frac{e^2}{8\pi^2\varepsilon}$. This depends only on e; therefore we use the chain rule:

$$\gamma_\Psi = \frac{1}{2}\frac{d\ln Z_2}{de}\frac{de}{d\ln\mu}$$

Using equation (66.1.1)

$$\gamma_\Psi = -\frac{e\varepsilon}{4}\frac{d\ln Z_2}{de}$$

Expanding:

$$\gamma_\Psi = -\frac{e\varepsilon}{4}\frac{d}{de}\left[-\frac{e^2}{8\pi^2\varepsilon}\right]$$

Taking the derivative:

$$\boxed{\gamma_\Psi = \frac{e^2}{16\pi^2}}$$

As for A, we proceed as before:

$$\gamma_A = -\frac{e\varepsilon}{4}\frac{d\ln Z_3}{de}$$

Using 62.24, expanding, and setting aside the finite portion (this will vanish anyway when $\varepsilon \to 0$):

$$\gamma_A = -\frac{e\varepsilon}{4}\frac{-e}{3\pi^2}\frac{1}{\varepsilon}$$

which is:

$$\boxed{\gamma_A = \frac{e^2}{12\pi^2}}$$

Srednicki 66.2. Compute the one-loop contributions to the anomalous dimensions of m, ϕ, and A^μ in scalar electrodynamics.

As before, we relate the bare fields to the renormalized fields to determine that $\phi_0 = Z_2^{1/2}\phi$, and further that $m_0 = Z_m^{1/2} Z_2^{-1/2} m$. This gives $\ln m_0 = \ln m + \frac{1}{2}\ln\left(\frac{Z_m}{Z_2}\right)$. Then,

$$\gamma_m = \frac{d\ln m}{d\ln\mu}$$

which is:

$$\gamma_m = \frac{d}{d\ln\mu}\left[\ln m_0 - \frac{1}{2}\ln\left(\frac{Z_m}{Z_2}\right)\right]$$

The bare parameters should be independent of μ, and so:

$$\gamma_m = -\frac{1}{2}\frac{d}{d\ln\mu}\ln(Z_m Z_2^{-1})$$

Expanding the logarithm about 0:

$$\gamma_m = -\frac{1}{2}\frac{d}{d\ln\mu}\left[\frac{\lambda}{8\pi^2\varepsilon} - \frac{3e^2}{8\pi^2\varepsilon}\right]$$

Using the chain rule, we have:

$$\gamma_m = -\frac{1}{2}\left[\frac{d}{de}\frac{de}{d\ln\mu} + \frac{d}{d\lambda}\frac{d\lambda}{d\ln\mu}\right]\left[\frac{\lambda}{8\pi^2\varepsilon} - \frac{3e^2}{8\pi^2\varepsilon}\right]$$

Equation (66.1.1) gives $\frac{de}{d\ln\mu} = -\frac{e\varepsilon}{2}$, and performing the analogous operation to equation 66.24 gives $\frac{d\lambda}{d\ln\mu} = -\lambda\varepsilon$. This gives:

$$-\frac{1}{2}\left[-\frac{e\varepsilon}{2}\frac{d}{de} - \lambda\varepsilon\frac{d}{d\lambda}\right]\left[\frac{\lambda}{8\pi^2\varepsilon} - \frac{3e^2}{8\pi^2\varepsilon}\right]$$

Doing the derivatives and simplifying, we have:

$$\boxed{\gamma_m = \frac{\lambda - 3e^2}{16\pi^2}}$$

Similarly, we have:

$$\gamma_\phi = \frac{1}{2}\frac{d\ln Z_\phi}{d\ln\mu}$$

Equaton 65.25 tells us that $Z_\phi = Z_2$ in this case. Thus,

$$\gamma_\phi = \frac{1}{2}\frac{d\ln Z_2}{d\ln\mu}$$

Using the chain rule:

$$\gamma_\phi = \frac{1}{2}\frac{d\ln Z_2}{de}\frac{de}{d\ln\mu}$$

Solving this, using equation (66.1.1):

$$\gamma_\phi = \frac{1}{2}\left[\frac{3e}{4\pi^2\varepsilon}\right]\left[-\frac{e\varepsilon}{2}\right]$$

Simplifying this, we have:

$$\boxed{\gamma_\phi = -\frac{3e^2}{16\pi^2}}$$

Now we repeat this for A, which has Z_3 as a coefficient. We have:

$$\gamma_A = \frac{1}{2}\frac{d\ln Z_3}{de}\frac{de}{d\ln\mu}$$

This gives:

$$\boxed{\gamma_A = \frac{e^2}{48\pi^2}}$$

Srednicki 66.3. Use the results of problem 62.2 to compute the anomalous dimenios of m and the beta function for e in spinor electrodynamics in R_ξ gauge. You should find that the results are independent of ξ.

From 62.3, we have $\Psi_0 = Z_2^{1/2}\Psi$, and so $m_0 = Z_2^{-1}Z_m m$. Taking the logarithm, we have

$$\ln m_0 = \ln m + \ln Z_m Z_2^{-1}$$

Taking the derivative, we have:

$$\frac{d\ln m_0}{d\ln\mu} = \frac{d\ln m}{d\ln\mu} + \frac{d\ln(Z_m/Z_2)}{d\ln\mu}$$

The bare fields are independent of μ, so:

$$\frac{d\ln m}{d\ln\mu} = -\frac{d}{d\ln\mu}\ln\left[Z_m/Z_2\right]$$

Now $\gamma_m = \frac{d\ln m}{d\ln\mu}$, so

$$\gamma_m = -\frac{d}{d\ln\mu}\left[\ln Z_m - \ln Z_2\right]$$

Using the chain rule and our results from problem 62.2, we have:

$$\gamma_m = -\frac{d}{de}\frac{de}{d\ln\mu}\left[\left(1 - \frac{e^2(3+\xi)}{8\pi^2\varepsilon}\right) - \ln\left(1 - \frac{e^2\xi}{8\pi^2\varepsilon}\right)\right]$$

From equation (66.1.1) (the derivation of which stands on its own), we have $\frac{de}{d\ln\mu} = -\frac{e\varepsilon}{2}$. Also recall that $ln(1+x) = x + \ldots$. Thus:

$$\gamma_m = -\left(-\frac{e\varepsilon}{2}\right)\frac{d}{de}\left[-\frac{e^2(3+\xi)}{8\pi^2\varepsilon} + \frac{e^2\xi}{8\pi^2\varepsilon}\right]$$

This gives:

$$\gamma_m = \frac{e\varepsilon}{2}\left[-\frac{3e}{4\pi^2\varepsilon}\right]$$

which is:

$$\boxed{\gamma_m = -\frac{3e^2}{8\pi^2}}$$

Now for the beta function. Matching bare fields with renormalized fields, we have $\Psi_0 = Z_2^{1/2}\Psi$, which implies $m_0 = Z_2^{-1}Z_m m$. Further, $A_0 = Z_3^{1/2}A$. These imply that:

$$e_0 = Z_2^{-1}Z_3^{-1/2}Z_1 e$$

Now we should shift the mass dimensionality off of e. We have $\mathcal{L} \sim \overline{\Psi}\partial\!\!\!/\Psi$; the ∂ has a mass dimensionality of 1, so the Ψ must have mass dimensionality of $(d-1)/2$. The last term in equation 62.1 reduces to $\mathcal{L} \sim \partial^\mu A^\nu \partial_\mu A_\nu$, so A has mass dimensionality of $(d-2)/2$. Now consider the term $\mathcal{L} \sim e\overline{\Psi}A\!\!\!/\Psi$. Using this, we see that in four dimensions, $[e] = 0$, and in six dimensions, $[e] = -1$. Thus, $[e] = \varepsilon/2$, where $\varepsilon = 4 - d$. Thus:

$$e_0 = Z_2^{-1}Z_3^{-1/2}Z_1 e\tilde{\mu}^{\varepsilon/2}$$

Using the result from problem 62.2, we have:

$$\ln Z_1 = -\frac{e^2\xi}{8\pi^2\varepsilon} + \dots$$

$$\ln Z_2^{-1} = \frac{e^2\xi}{8\pi^2\varepsilon} + \dots$$

$$\ln Z_3^{-1/2} = \frac{e^2}{12\pi^2\varepsilon} + \dots$$

This gives:

$$\ln Z_1 Z_2^{-1} Z_3^{-1/2} = E = \sum_{i=1}^{\infty} \frac{E_i}{\varepsilon^i} = \frac{e^2}{12\pi^2\varepsilon} + \dots$$

Now we have:

$$\ln e_0 = E + \ln e + \frac{\varepsilon}{2}\ln\mu$$

Taking the derivative with respect to $\ln\mu$, and using the chain rule:

$$0 = \frac{\partial E}{\partial e}\frac{\partial e}{\partial \ln\mu} + \frac{\partial \ln e}{\partial \ln\mu} + \varepsilon/2$$

Take the derivative in the second term:

$$0 = \frac{\partial E}{\partial e}\frac{\partial e}{\partial \ln\mu} + \frac{1}{\varepsilon}\frac{\partial e}{\partial \ln\mu} + \varepsilon/2$$

Now we factor:

$$0 = \left(e\frac{\partial E}{\partial e} + 1\right)\frac{\partial e}{\partial \ln\mu} + \frac{e\varepsilon}{2}$$

Now we use our usual "physical reasoning" trick: a renormalizabel theory should be well defined as $\varepsilon \to 0$, so we can set aside the E term. Then:

$$\frac{de}{d\ln\mu} = -\frac{e\varepsilon}{2} + \beta(e)$$

This gives:

$$0 = \left(e\frac{\partial E}{\partial e} + 1\right)\left(-\frac{e\varepsilon}{2} + \beta(e)\right) + \frac{e\varepsilon}{2}$$

Now we distribute, and match up the terms with no εs. We find:

$$e\frac{\partial E_1}{\partial e}\frac{1}{\varepsilon} \cdot \left(-\frac{e\varepsilon}{2}\right) + \beta(e) = 0$$

This gives:

$$\boxed{\beta(e) = \frac{e^3}{12\pi^2}}$$

Srednicki 66.4. The value of $\alpha(M_W)$. The solution of equation 66.12 is:

$$\frac{1}{\alpha(M_W)} = \frac{1}{\alpha(\mu)} - \frac{2}{3\pi}\sum_i Q_i^2 \ln\left(M_W/\mu\right)$$

where the sum is over all quarks and leptons (each color of quark counts separately), and we have chosen the $W^\pm$ boson mass M_W as a reference scale. We can define a different renormalization scheme, *modified decoupling subtraction* or $\overline{\text{DS}}$, where we imagine integrating out a field when μ is below its mass. In this scheme, equation 66.30 becomes:

$$\frac{1}{\alpha(M_W)} = \frac{1}{\alpha_\mu} - \frac{2}{3\pi}\sum_i Q_i^2 \ln\left[M_W/\min(m_i,\mu)\right]$$

where the sum is now over all quarks and leptons with mass less that M_W. For $\mu < m_e$, the $\overline{\text{DS}}$, scheme coincides with the OS scheme, and we have

$$\frac{1}{\alpha(M_W)} = \frac{1}{\alpha} - \frac{2}{3\pi}\sum_i Q_i^2 \ln\left(M_W/m_i\right)$$

where $\alpha = 1/137.036$ is the fine-structure constant in the OS scheme. Using $m_\mu = m_d = m_s \sim 300$ MeV for the light quark masses (because quarks should be replaced by hadrons at lower energies), and other quark and lepton masses from sections 83 and 88, compute $\alpha(M_W)$.

It's just a matter of plugging into the formula. We have:

$$\frac{1}{\alpha(M_w)} = 137.036 - \frac{2}{3\pi}\left\{(\pm 1)^2 \ln\left[\frac{M_W^3}{0.511\cdot 105.7\cdot 1777}\right] + 3\cdot\left(\frac{2}{3}\right)^2 \ln\left[\frac{M_W^3}{300\cdot 1300\cdot 174000}\right] +\right.$$

$$3 \cdot \left(\frac{1}{3}\right)^2 \ln\left[\frac{M_W^3}{300 \cdot 300 \cdot 4300}\right]\Bigg\}$$

which is:

$$\frac{1}{\alpha(M_w)} = 128.7$$