Srednicki Chapter 64

QFT Problems & Solutions

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Srednicki 64.1. Let the wave packet be $f(\vec{p}) \propto \exp\left(-a^2\vec{p}^2/2\right) Y_{\ell m}(\hat{p})$, where $Y_{\ell m}(\hat{p})$ is a spherical harmonic. Find the contribution of the orbital angular momentum to the magnetic moment.

The key point is that we want the *orbital* angular momentum's contribution to the magnetic moment, not the entire magnetic moment.

This would be easy to do if we had an $\vec{L}$ term and an $\vec{S}$ term in our answer. Recall that we have $L_z = x\partial_y - y\partial_x$. A little trial and error shows that this is easiest to get if we choose the following vector potential:

$$\vec{A} = \frac{B}{2}\langle -y, x, 0\rangle$$

Now recall that $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. This gives us that $F^{12} = -F^{21} = B$, and all other components of F are zero.

Now we plug in everything we've found into 64.3, drop the time integral to move from the action to the Lagrangian, and introduce a negative sign to move from the Lagrangian to the Hamiltonian.

$$H_1 = -\frac{eB}{2}\int d^3x \overline{\Psi}\left[-y\gamma^1 + x\gamma^2 + \frac{\alpha}{\pi m}S^{12}\right]\Psi$$

Using equation 64.7 (definition of the magnetic moment), we can identify:

$$\mu = \frac{e}{2}\int d^3x\ \langle e|\overline{\Psi}\left[-y\gamma^1 + x\gamma^2 + \frac{\alpha}{\pi m}S^{12}\right]\Psi|e\rangle$$

Now we can use 64.4 to get rid of the bracket and the Dirac fields, replacing them instead with just a spinor:

$$\mu = \frac{e}{2}\int d^3x\widetilde{dp}\widetilde{dp'}\ f^*(p')\langle 0|b_+(p')\overline{\Psi}\left[-y\gamma^1 + x\gamma^2 + \frac{\alpha}{\pi m}S^{12}\right]\Psi b_+^\dagger(p)f(p)|0\rangle$$

Now the vacuum will be affected only by things that can be decomposed into creation operators, which will be the Dirac field, and the creation operator itself. We can therefore use equation 64.8 – note that we have to be careful with where we place the spinors, since we don't want to resort to index notation:

$$\mu = \frac{e}{2}\int d^3x\ \widetilde{dp}\ \widetilde{dp'}\ f^*(p')\overline{u}_+(p')\left[-y\gamma^1 + x\gamma^2 + \frac{\alpha}{\pi m}S^{12}\right]u_+(p)e^{i(p-p')x}f(p)$$

Next we rewrite $x = -i\partial_{px}$ and $y = -i\partial_{py}$ acting on the exponential; we then integrate by parts (which switches the sign back) and puts the derivative acting on the $u_+(p)f(p)$. Then:

$$\mu = \frac{e}{2}\int d^3x\ \widetilde{dp}\ \widetilde{dp'}\ e^{i(p-p')x}\ f^*(p')\overline{u}_+(p')\left[-i\gamma^1\partial_{py} + i\gamma^2\partial_{px} + \frac{\alpha}{\pi m}S^{12}\right]u_+(p)f(p)$$

Now we can do the integral over d^3x:

$$\mu = \frac{(2\pi)^3 e}{2}\int \widetilde{dp}\ \widetilde{dp'}\ \delta^3(p-p')\ f^*(p')\overline{u}_+(p')\left[-i\gamma^1\partial_{py} + i\gamma^2\partial_{px} + \frac{\alpha}{\pi m}S^{12}\right]u_+(p)f(p)$$

Now we do the p' integral:

$$\mu = \frac{e}{4\omega}\int \widetilde{dp}\ f^*(p)\overline{u}_+(p)\left[-i\gamma^1\partial_{py} + i\gamma^2\partial_{px} + \frac{\alpha}{\pi m}S^{12}\right]u_+(p)f(p)$$

Next we use the product rule. Note that S^{12} is a matrix, and so will not act on a function. Then:

$$\mu = \frac{e}{4\omega}\int \widetilde{dp}\ f^*(p)\overline{u}_+(p)\left[-i\gamma^1 f(p)\partial_{py}u_+(p) + i\gamma^2 f(p)\partial_{px}u_+(p) - i\gamma^1 u_+(p)\partial_{py}f(p) + \right.$$

$$\left. i\gamma^2 u_+(p)\partial_{px}f(p) + \frac{\alpha}{\pi m}f(p)S^{12}u_+(p)\right]$$

Now f(p) is still peaked around $p=0$, so we can still use equation 64.12 and 64.13. This can be applied directly to the second term; to apply it to the first term, we get an additional minus sign from $S^{21} = -S^{12}$. Thus:

$$\mu = \frac{e}{4\omega}\int \widetilde{dp}\ f^*(p)\left[\frac{2}{m}f(p)\overline{u}_+(0)S^{12}u_+(0) - i\overline{u}_+(0)\gamma^1 u_+(p)\partial_{py}f(p) + i\overline{u}_+(0)\gamma^2 u_+(p)\partial_{px}f(p)\right.$$

$$\left. + \frac{\alpha}{\pi m}f(p)\overline{u}_+(p)S^{12}u_+(p)\right]$$

In the last term we expanding around $p=0$ (since the expression is strongly peaked around p=0), finding that u(p) = u(0). We can then combine the first and last terms:

$$\mu = \frac{e}{4\omega}\int \widetilde{dp}\ f^*(p)\left[\frac{1}{m}\left(2 + \frac{\alpha}{\pi}\right)f(p)\overline{u}_+(0)S^{12}u_+(0) - i\overline{u}_+(0)\gamma^1 u_+(p)\partial_{py}f(p) + i\overline{u}_+(0)\gamma^2 u_+(p)\partial_{px}f(p)\right]$$

Now recall $\overline{u}_+(p)\gamma^i u_+(p) = 2p^i$. Then:

$$\mu = \frac{e}{4\omega}\int \widetilde{dp}\ f^*(p)\left[\frac{1}{m}\left(2 + \frac{\alpha}{\pi}\right)f(p)\overline{u}_+(0)S^{12}u_+(0) - 2ip^x\partial_{py}f(p) + i2p^y\partial_{px}f(p)\right]$$

Now we can factor a bit:

$$\mu = \frac{e}{2\omega}\int \widetilde{dp}\; f^*(p)\left[\frac{1}{m}\left(1+\frac{\alpha}{2\pi}\right)f(p)\overline{u}_+(0)S^{12}u_+(0) - i\left(p^x\partial_{py} - p^y\partial_{px}\right)f(p)\right]$$

Now recall that $\hat{L} = r \times p$, which in the momentum basis is $\hat{L} = i\partial_{px} \times p$ (the derivative does not act on the p). Then $\hat{L}_z = ip_y\partial_{px} - ip_x\partial_{py}$. Then:

$$\mu = \frac{e}{2\omega}\int \widetilde{dp}\; f^*(p)\left[\frac{1}{m}\left(1+\frac{\alpha}{2\pi}\right)f(p)\overline{u}_+(0)S^{12}u_+(0) + L_z f(p)\right]$$

Further, notice that $S_z = S^{12}$:

$$\mu = \frac{e}{2\omega}\int \widetilde{dp}\; f^*(p)\left[\frac{1}{m}\left(1+\frac{\alpha}{2\pi}\right)f(p)\overline{u}_+(0)S_z u_+(0) + L_z f(p)\right] \tag{64.1.1}$$

L_z acting on the spherical harmonic, will give m_ℓ. Similarly, S_z acting on the spherical harmonic will give m_s. Then:

$$\mu = \frac{e}{2\omega}\int \widetilde{dp}\; f^*(p)\left[\frac{1}{m}\left(1+\frac{\alpha}{2\pi}\right)f(p)\overline{u}_+(0)m_s u_+(0) + m_\ell f(p)\right]$$

Now $\overline{u}_+(0)u_+(0) = 2m$, so:

$$\mu = \frac{e}{2\omega}\int \widetilde{dp}\; f^*(p)\left[2m_s\left(1+\frac{\alpha}{2\pi}\right)f(p) + m_\ell f(p)\right]$$

Now the wave packet is normalized; this allows us to do the integral as in equation 64.14 (notice in the text that the integral involves swapping ω for m). Nothing changes since the spherical harmonics are also normalized in the same way. Finally, recall that for an electron, $m_s = 1/2$. So:

$$\mu = \frac{e}{m}\left[\frac{1}{2}\left(1+\frac{\alpha}{2\pi}\right) + \frac{m_\ell}{2}\right]$$

We see that the first term corresponds exactly to equation 64.16, so the second term must be due to the orbital angular momentum. This is also apparent by noticing that every term in equation (64.1.1) has an operator, and the last term has the L_z operator. Thus, the contribution to the orbital angular momentum in this case is:

$$\mu_\ell = \frac{e m_\ell}{2m}$$

For an electron, $m_\ell = \pm\frac{1}{2}$.

Note: this is a wonderful and worthwhile problem, but it is not clear to me that it is obvious to switch gauges. A hint to that effect would be helpful. Normally I would try to provide a more intuitive solution than Srednicki; however in this case, the alternative involves trying to find a general formula for the derivative of spherical harmonics in Cartesian Coordinates. I do not believe this is possible, but even if it were, I personally would prefer to gouge my own eyes out with a meat cleaver.