Srednicki Chapter 62

QFT Problems & Solutions

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Srednicki 62.1. Show that adding a gauge fixing term $-\frac{1}{2}\xi^{-1}(\partial^\mu A_\mu)^2$ to $\mathcal{L}$ results in equation 62.9 as the photon propagator. Explain why $\xi = 0$ corresponds to Lorentz gauge $\partial^\mu A_\mu = 0$.

The term in question is:

$$\mathcal{L}_{GF} = -\frac{1}{2}\xi^{-1}(\partial^\mu A_\mu)^2$$

This term will contribute the following to the action:

$$S_{GF} = -\frac{1}{2}\xi^{-1}\int d^4x\ \partial^\mu A_\mu(x)\partial^\nu A_\nu(x)$$

Next we take the Fourier transform, which is:

$$f(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx}\widetilde{f}(k)$$

Thus:

$$S_{GF} = \frac{1}{2}\xi^{-1}\int d^4x \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4} e^{-ikx}e^{-ik'x}\ k^\mu k'^\nu \tilde{A}_\mu(x)\tilde{A}_\nu(x)$$

Next we do the x-integral:

$$S_{GF} = \frac{1}{2}\xi^{-1}\int d^4k \frac{d^4k'}{(2\pi)^4}\delta^4(k+k')\ k^\mu k'^\nu \tilde{A}_\mu(x)\tilde{A}_\nu(x)$$

Next we do the k' integral:

$$S_{GF} = -\frac{1}{2}\xi^{-1}\int \frac{d^4k}{(2\pi)^4}\ k^\mu k^\nu \tilde{A}_\mu(k)\tilde{A}_\nu(-k)$$

Now let's write down the entire action of section 57, with this new term added:

$$S = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left[-\tilde{A}_\mu(k)\left(k^2 g^{\mu\nu} - (1-\xi^{-1})k^\mu k^\nu\right)\tilde{A}_\nu(-k) + \tilde{J}^\mu(k)\tilde{A}_\mu(-k) + \tilde{J}^\mu(-k)\tilde{A}_\mu(k)\right] \tag{62.1.1}$$

Next we change the path integration variable to:

$$\tilde{\chi}(k) = \tilde{A}(k) - \left[k^2 g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) k^\mu k^\nu\right]^{-1} \tilde{J}(k)$$

As usual, we need to invert this matrix. Let's "guess" the following form (second brackets), then check it:

$$\left[k^2 g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) k^\mu k^\nu\right] \left[g_{\nu\rho}/k^2 - (1-\xi)\frac{k_\nu k_\rho}{k^2 k^2}\right] \stackrel{?}{=} \delta^\mu_{\ \rho}$$

Multiplying:

$$g^{\mu\nu} g_{\nu\rho} - g^{\mu\nu}(1-\xi)\frac{k_\nu k_\rho}{k^2} - \left(1 - \frac{1}{\xi}\right)\frac{k^\mu k^\nu}{k^2} g_{\nu\rho} + \left(1 - \frac{1}{\xi}\right)(1-\xi)\frac{k^\mu k^\nu k_\nu k_\rho}{k^2 k^2} \stackrel{?}{=} \delta^\mu_{\ \rho}$$

Using the metric:

$$g^\mu_{\ \rho} - (1-\xi)\frac{k^\mu k_\rho}{k^2} - \left(1 - \frac{1}{\xi}\right)\frac{k^\mu k_\rho}{k^2} + \left(1 - \frac{1}{\xi}\right)(1-\xi)\frac{k^\mu k_\rho}{k^2} \stackrel{?}{=} \delta^\mu_{\ \rho}$$

Simplifying:

$$g^\mu_{\ \rho} - \frac{k^\mu k_\rho}{k^2}\left[(1-\xi) + \left(1 - \frac{1}{\xi}\right) - \left(1 - \frac{1}{\xi}\right)(1-\xi)\right] \stackrel{?}{=} \delta^\mu_{\ \rho}$$

The term in brackets vanishes, leaving:

$$g^\mu_{\ \rho} \stackrel{\checkmark}{=} \delta^\mu_{\ \rho}$$

Thus, our path integration variable becomes:

$$\tilde{\chi}^\mu(k) = \tilde{A}^\mu(k) - \left[g^{\mu\nu}/k^2 - (1-\xi)\frac{k^\mu k^\nu}{k^2 k^2}\right] \tilde{J}_\nu(k)$$

With this substitution, our action becomes:

$$S_0 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left\{\tilde{J}(k)\left[g_{\nu\rho}/k^2 - (1-\xi)\frac{k_\nu k_\rho}{k^2 k^2}\right]\tilde{J}(-k) - \tilde{\chi}(k)\left[k^2 g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) k^\mu k^\nu\right]\tilde{\chi}(-k)\right\}$$

Following the procedure in chapter 8, we perform the integral over χ, which yields a factor of one. Further, we do the usual epsilon trick. Thus:

$$Z(J) = \exp\left[\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\tilde{J}^\mu(k)\frac{1}{k^2 - i\varepsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\tilde{J}^\nu(-k)\right]$$

We read off the propagator from this:

$$\tilde{\Delta}_{\mu\nu}(k) = \frac{1}{k^2 - i\varepsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)$$

which is equation 62.9.

As to how this is Lorentz Gauge, we return to the full path integral (62.1.1), which has a factor of ξ^{-1}. Thus, this will vanish more and more rapidly, which vanishes by the Riemann-Lebesque Lemma. To avoid a vanishing path integral, we need this term to be set to zero through another means, namely through $k^\mu \tilde{A}_\mu(k) = 0$, which in position space would be $\partial^\mu A_\mu(k) = 0$. This is Lorentz Gauge.

Note: Srednicki's solution here is even more wrong than unusual. He states "Since $P^{\mu\nu}(k)$ and $k^\mu k^\nu/k^2$ are orthogonal projection matrices, the propagator is $(1/k^2)[P^{\mu\nu} + \xi k^\mu k^\nu/k^2]$." *It is true that these are orthogonal projection matrices, and it is therefore possible to take the inverse of both terms separately. But, precisely because they are projection matrices, both matrices are non-invertible. With respect to $P^{\mu\nu}$, this difficulty was addressed in chapter 57. But $k^\mu k^\nu/k^2$ is non invertible. The argument presented in chapter 57 does not apply here. In fact, that argument proves that anything that will survive this projection matrix will not contribute to the integral, and so the matrix maps everything onto 0, destroying the information. There is no way to reverse this.*

However, this is not a problem: the entire matrix $k^2[P^{\mu\nu} + \xi^{-1}k^\mu k^\nu/k^2]$ is not a projection matrix, and is completely invertible, as shown above. The only exception is when $\xi = 0$, of course – that's the special case that Srednicki treated in chapter 57. In some sense, then, Srednicki did the special case, leaving us with the easier (but more important) general case. We can therefore appreciate the importance of this problem for the conceptual completeness of our study of QFT. Further, the result (equation 62.9) is very important; we will do something similar for gluons in chapter 72 with little discussion, as that argument is analogous to this one.

Srednicki 62.2. Find the coefficients of e^2/ε of $Z_{1,2,3,m}$ in R_ξ gauge. In particular, show that $Z_1 = Z_2 = 1 + O(e^4)$ in Lorenz Gauge.

Note: apparently Hendrik Lorentz (who you've probably heard of) and Ludvig Lorenz (probably not) were contemporaneous physicists with similar research interests. The gauge condition Srednicki specifies correctly refers to Lorenz, not Lorentz.

Let's be a little bit clear. Srednicki already calculated these, but he did so in Feynman Gauge, ie with $\xi = 1$. We need to repeat this with a general ξ.

So what does $\xi \neq 1$ change? As we showed in problem 62.1, the corresponding term in the Lagrangian affects the free propagator only; it does not introduce an interaction term. Therefore, only the photon propagator changes.

Note that figure 62.1 does not include a photon propagator. Thus, Z_3 is unchanged, and we can read off the coefficient to e^2/ε from equation 62.24 (ignoring the finite terms):

$$\boxed{Z_3 = 1 - \frac{e^2}{6\pi^2\varepsilon}}$$

Now we turn to figure 62.2. Assessing the value of this still yields equation 62.28:

$$i\Sigma(\not{p}) = (iZ_1 e)^2 \left(\frac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4} \left[\gamma^\nu \tilde{S}(\not{p}+\not{\ell})\gamma^\mu\right] \tilde{\Delta}_{\mu\nu}(\ell) - i(Z_2-1)\not{p} - i(Z_m-1)m$$

However, in this gauge the photon propagator is given by equation 62.9 rather than 62.29:

$$\tilde{\Delta}_{\mu\nu}(\ell) = \frac{g_{\mu\nu} + (\xi-1)\ell_\mu\ell_\nu/\ell^2}{(\ell^2 + m_\gamma^2 - i\varepsilon)}$$

As discussed in the chapter, we set $Z_1 = 1$, anticipating that it will have no first-order terms in ε. Then, with $\tilde{S}$ as given below equation 62.12, we have:

$$i\Sigma(\not{p}) = e^2 \int \frac{d^4\ell}{(2\pi)^4} \left[\gamma^\nu \left(\frac{-\not{p} - \not{\ell} + m}{(p+\ell)^2 + m^2 - \varepsilon}\right) \gamma^\mu \left(\frac{g_{\mu\nu} + (\xi-1)\ell_\mu\ell_\nu/\ell^2}{\ell^2 + m_\gamma^2}\right)\right] - i(Z_2-1)\not{p} - i(Z_m-1)m$$

Now we can drop all the infinitesimal terms in the denominator, since we don't have to worry about the denominator vanishing. This gives:

$$i\Sigma(\not{p}) = i\Sigma_{ch}(\not{p}) + e^2 \int \frac{d^4\ell}{(2\pi)^4} \left[\gamma^\nu \left(\frac{-\not{p} - \not{\ell} + m}{(p+\ell)^2 + m^2}\right) \gamma^\mu \left(\frac{(\xi-1)\ell_\mu\ell_\nu/\ell^2}{\ell^2}\right)\right]$$

where $\Sigma_{ch}(\not{p})$ refers to the Σ that Srednicki worked out in the chapter. Then:

$$\Delta\Sigma(\not{p}) = e^2 \int \frac{d^4\ell}{(2\pi)^4} \left[\gamma^\nu \left(\frac{-\not{p} - \not{\ell} + m}{(p+\ell)^2}\right) \gamma^\mu \left(\frac{(\xi-1)\ell_\mu\ell_\nu/\ell^2}{\ell^2}\right)\right]$$

which is:

$$\Delta\Sigma(\not{p}) = e^2(\xi-1) \int \frac{d^4\ell}{(2\pi)^4} \frac{\not{\ell}(-\not{p} - \not{\ell} + m)\not{\ell}}{[(p+\ell)^2 + m^2][\ell^2][\ell^2]}$$

Now we use Feynman's Formula to combine the denominator.

$$\Delta\Sigma(\not{p}) = 2e^2(\xi-1) \int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{\not{\ell}(-\not{p} - \not{\ell} + m)\not{\ell}}{[((p+\ell)^2 + m^2)x_1 + \ell^2 x_2 + \ell^2(1 - x_1 - x_2)]^3}$$

Simplifying this, we arrive at:

$$\Delta\Sigma(\not{p}) = 2e^2(\xi-1) \int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{\not{\ell}(-\not{p} - \not{\ell} + m)\not{\ell}}{[p^2 x_1 + 2(\ell \cdot p)x_1 + m^2 x_1 + \ell^2]^3}$$

We can do the x_2 integral:

$$\Delta\Sigma(\not{p}) = 2e^2(\xi-1) \int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dx_1 \frac{\not{\ell}(-\not{p} - \not{\ell} + m)\not{\ell}(1-x_1)}{[p^2 x_1 + 2(\ell \cdot p)x_1 + m^2 x_1 + \ell^2]^3}$$

Now we define $q = \ell + x_1 p$, and $D = m^2 x_1 + p^2 x_1(1 - x_1)$. This gives:

$$\Delta\Sigma(\not{p}) = 2e^2(\xi-1) \int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dx \frac{\not{\ell}(-\not{p} - \not{\ell} + m)\not{\ell}(1-x)}{(q^2 + D)^3}$$

Next we use $\ell = q - xp$ to write this as:

$$\Delta\Sigma(\not{p}) = 2e^2(\xi - 1)\int \frac{d^4q}{(2\pi)^4}\int_0^1 dx \frac{(\not{q} - x\not{p})(-\not{p} - \not{\ell} + m)(\not{q} - x\not{p})(1-x)}{(q^2+D)^3}$$

Now recall that only terms even in q integrate to something nonzero. The zero-order terms can also be neglected, because something with $O(q^{-6})$, after four integrals, will still be of order $O(q^{-2})$, which will not diverge.

$$\Delta\Sigma(\not{p}) = 2e^2(\xi - 1)\int \frac{d^4q}{(2\pi)^4}\int_0^1 dx \frac{N(1-x)}{(q^2+D)^3} + \text{ (finite)}$$

where

$$N = -\not{q}\not{p}\not{q} + \not{q}\not{q}x\not{p} + \not{q}x\not{p}\not{q} - x\not{p}\not{q}\not{q} + m\not{q}\not{q}$$

Now we can use equation 62.18:

$$N = -\frac{1}{4}g^{\mu\nu}q^2[-\gamma_\mu\not{p}\gamma_\nu + \gamma_\mu\gamma_n ux\not{p} + \gamma_\mu x\not{p}\gamma_\nu + x\not{p}\gamma_\mu\gamma_\nu + m\gamma_\mu\gamma_\nu]$$

Contracting this and using equation 47.18 and 47.19:

$$N = \left[-\frac{1}{2}\not{p}(1+3x)m\right]q^2$$

Thus (neglecting to write the finite term):

$$\Delta\Sigma(\not{p}) = -2e^2(\xi - 1)\int \frac{d^4q}{(2\pi)^4}\int_0^1 dx \frac{[-\frac{1}{2}\not{p}(1+3x)+m](1-x)q^2}{(q^2+D)^3}$$

Now we take a Wick Rotation and use equation 14.27:

$$\Delta\Sigma(\not{p}) = -2ie^2(\xi - 1)\int_0^1 dx \frac{[-\frac{1}{2}\not{p}(1+3x)+m](1-x)}{8\pi^2\varepsilon}$$

Doing this last integral and simplifying, we have:

$$\Delta\Sigma(\not{p}) = \frac{-ie^2(\xi-1)}{8\pi^2\varepsilon}(\not{p}+m)$$

So, we need to add $\frac{-e^2(\xi-1)}{8\pi^2\varepsilon}$ to Srednicki's answer for Z_1, and $\frac{-e^2(\xi-1)}{8\pi^2\varepsilon}$ to Z_m. This gives:

$$\boxed{Z_2 = 1 - \frac{e^2\xi}{8\pi^2\varepsilon}}$$

$$\boxed{Z_m = 1 - \frac{e^2(3+\xi)}{8\pi^2\varepsilon}}$$

Note that if $\xi = 1$, we recover Srednicki's answer, as expected.

Finally, we have to get Z_1 using the vertex correction. Equation 62.40 is still good:

$$iV = (ie)^3\left(\frac{1}{i}\right)^3\int \frac{d^4\ell}{(2\pi)^4}\left[\gamma^\rho\tilde{S}(p'+\ell)\gamma^\mu\tilde{S}(p+\ell)\gamma^\nu\right]\tilde{\Delta}_{\nu\rho}(\ell)$$

Now 62.9 is:

$$\tilde{\Delta}^{\mu\nu} = \frac{g^{\mu\nu} + (\xi - 1)\ell^\mu \ell^\nu/\ell^2}{\ell^2 - i\varepsilon}$$

Srednicki did this first term, so we keep only the second term:

$$\Delta V = -ie^3 \int \frac{d^4\ell}{(2\pi)^4} \left[\gamma^\rho \frac{-\not{p}' - \not{\ell} + m}{(p' + \ell)^2 + m^2 - i\varepsilon} \gamma^\mu \frac{-\not{p} - \not{\ell} + m}{(p + \ell)^2 + m^2 - i\varepsilon} \gamma^\nu\right] \frac{(\xi - 1)\ell_\nu \ell_\rho}{\ell^2(\ell^2 - i\varepsilon)}$$

Now if this term is to contribute, ℓ must be nonzero, so we can drop the infinitesimals in the denominator:

$$\Delta V = -ie^3(\xi - 1) \int \frac{d^4\ell}{(2\pi)^4} \left[\not{\ell} \frac{-\not{p}' - \not{\ell} + m}{(p' + \ell)^2 + m^2} \gamma^\mu \frac{-\not{p} - \not{\ell} + m}{(p + \ell)^2 + m^2} \not{\ell}\right] \frac{1}{\ell^2 \ell^2}$$

Now we define $N^\mu = \not{\ell}(-\not{p}' - \not{\ell} + m)\gamma^\mu(-\not{p} - \not{\ell} + m)\not{\ell}$. Also, we use Feynman's formula for the denominators. Thus:

$$\Delta V = -6ie^3(\xi - 1) \int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \times$$

$$\left[\frac{N^\mu}{\{[(p' + \ell)^2 + m^2]\, x_1 + [(p + \ell)^2 + m^2]\, x_2 + \ell^2 x_3 + \ell^2(1 - x_1 - x_2 - x_3)\}^4}\right]$$

We define $q = \ell + x_1 p + x_2 p'$. Then:

$$N^\mu = (\not{q} - x_1\not{p} - x_2\not{p}')(-\not{p}' - \not{q} + x_1\not{p} + x_2\not{p}' + m)\gamma^\mu(-\not{p} - \not{q} + x_1\not{p} + x_2\not{p}')(\not{q} - x_1\not{p} - x_2\not{p}')$$

Similarly, we simplify the denominator to $(q^2 + D)^4$, where D is:

$$D = -\ell^2 - 2x_1x_2pp' - x_1^2p^2 - x_2^2p'^2 + p'^2x_1 + m^2x_1 + p^2x_2 + m^2x_2$$

We can also do the x_3 integral. Thus:

$$\Delta V = -6ie^3(\xi - 1) \int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{(1 - x_1 - x_2)N^\mu}{(q^2 + D)^4}$$

Now in the denominator, the odd terms vanish, and the terms of $O(q^3)$ or less are not divergent. Thus $N^\mu = \not{q}\not{q}\gamma^\mu\not{q}\not{q}$ is the only term that contributes to the divergence, so:

$$\Delta V = -6ie^3(\xi - 1) \int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{(1 - x_1 - x_2)\not{q}\not{q}\gamma^\mu\not{q}\not{q}}{(q^2 + D)^4} + \text{ (finite)}$$

To simplify this mess in the numerator, we use the result from problem 14.3 (*I cannot resist complaining that Srednicki requires us to use the solution to this problem without giving the answer to that problem. Hopefully everyone kept their homework from 48 chapters ago.....*).

$$\Delta V = -6ie^3(\xi - 1) \int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{q^4\gamma^\mu(1 - x_1 - x_2)}{(q^2 + D)^4}$$

Now recall that D is loaded with x_1s and x_2s, so we cannot (easily) do those integrals yet. On the other hand, we can easily do a Wick Rotation and use equation 14.27;

$$\Delta V = 6e^3(\xi - 1)\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \gamma^\mu (1 - x_1 - x_2)\left(\frac{2}{\varepsilon} + (\text{const})\right)\frac{1}{(4\pi)^2 1!}\left(1 - \frac{\varepsilon}{2}D\right)$$

We only want the divergent terms, so most of these terms can be absorbed into the (constant) term which we neglected before. We're left with:

$$\Delta V = \frac{12e^3(\xi - 1)}{(4\pi)^2\varepsilon}\gamma^\mu \int_0^1 dx_1 \int_0^{1-x_1} dx_2 (1 - x_1 - x_2)$$

Doing these easy integrals, we're left with:

$$\Delta V = \frac{e^3(\xi - 1)}{8\pi^2\varepsilon}\gamma^\mu$$

So, equation 60.39 shows that Z_1 needs to include a term of $-\frac{e^2(\xi-1)}{8\pi^2\varepsilon}$ to cancel this. Adding this to Srednicki's solution (equation 62.50), we have:

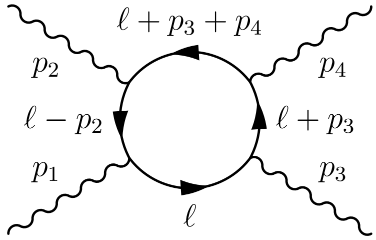
$$\boxed{Z_1 = 1 - \frac{e^2\xi}{8\pi^2\varepsilon}}$$

Just as Srednicki promised, we have $Z_1 = Z_2 = 1 + O(e^4)$ if $\xi = 0$, which is Lorenz gauge. Better yet, $Z_1 = Z_2$ in general R_ξ gauge.

Note: Notice that our answer is independent of p, p', as it must be when we're considering the divergent terms. We could therefore have made this a bit easier for ourselves by setting $p = p' = 0$. Perhaps I was unjustly harsh in complaining about the lack of solution to problem 14.3(b).

Srednicki 62.3. Consider the six one-loop diagrams with four external photons (and no external fermions). Show that, even though each diagram is logarithmically divergent, their sum is finite. Use gauge invariance to explain why this must be the case.

The diagram in question is:



Note that there are five permutations of this, the five permutations of 2, 3, and 4.

We can assess the value of this. Don't forget the overall minus sign due to the fermion loop; further, consider all particles as outgoing for notational simplicity (it doesn't matter since we will take the external momenta to 0 in the next step):

$$iV = -\int \frac{d^4\ell}{(2\pi)^4} \varepsilon_1^\mu(k_1)\varepsilon_2^\nu(k_2)\varepsilon_3^\sigma(k_3)\varepsilon_4^\rho(k_4) \frac{-i(-\not\ell + m)}{\ell^2 + m^2 - i\varepsilon}(ie\gamma_\nu)\frac{-i(-\not\ell + \not p_2 + m)}{(\ell - p_2)^2 + m^2 - i\varepsilon}(ie\gamma_\mu)$$

$$\frac{-i(-\not\ell - \not p_3 - \not p_4 + m)}{(\ell + p_3 + p_4)^2 + m^2 - i\varepsilon}(ie\gamma_\rho)\frac{-i(-\not\ell - \not p_3 + m)}{(\ell + p_3)^2 + m^2 - i\varepsilon}(ie\gamma_\sigma) + \text{ (5 perms)}$$

Now we need the momentum in the loop to be nonzero, which means that the denominator cannot vanish. Thus, we drop the infinitesimals. Further, we only want divergences, so we can set the external momenta to zero. Finally, we simplify:

$$iV = -e^4\varepsilon_1^\mu\varepsilon_2^\nu\varepsilon_3^\sigma\varepsilon_4^\rho \int \frac{d^4\ell}{(2\pi)^4}\left(\frac{-\not\ell + m}{\ell^2 + m^2}\right)\gamma_\nu\left(\frac{-\not\ell + m}{\ell^2 + m^2}\right)\gamma_\mu\left(\frac{-\not\ell + m}{\ell^2 + m^2}\right)\gamma_\rho\left(\frac{-\not\ell + m}{\ell^2 + m^2}\right)\gamma_\sigma + \text{ (5 perms)}$$

In the denominator we have ℓ^8. The terms in the numerator of $O(\ell^4)$ will, after four integrals, be logarithmically divergent. The other terms will converge. Thus, we can rop the mass terms in the numerator. Further, ε_i^μ is a number, so we can move them around:

$$iV = -e^4\int \frac{d^4\ell}{(2\pi)^4}\frac{\not\ell\not\varepsilon_2\not\ell\not\varepsilon_1\not\ell\not\varepsilon_4\not\ell\not\varepsilon_3}{(\ell^2+m^2)^4} + \text{ (const)} + \text{(5 perms)}$$

Let's drop this constant term, since we are only interested in the divergent component. Next, let's use the result from problem 14.3(b). (*This time, I really can complain about the lack of solution to this problem being provided in the book*):

$$iV = -e^4\int \frac{d^4\ell}{(2\pi)^4}\ell^4\left[\frac{\gamma^\mu\not\varepsilon_2\gamma^\nu\not\varepsilon_1\gamma_\nu\not\varepsilon_4\gamma_\mu\not\varepsilon_3}{(\ell^2+m^2)^4} + \frac{\gamma^\mu\not\varepsilon_2\gamma^\nu\not\varepsilon_1\gamma_\mu\not\varepsilon_4\gamma_\nu\not\varepsilon_3}{(\ell^2+m^2)^2} + \frac{\gamma^\mu\not\varepsilon\gamma_\mu\not\varepsilon_1\gamma^\nu\not\varepsilon_4\gamma_\nu\not\varepsilon_3}{(\ell^2+m^2)^4}\right] + \text{(5 perms)}$$

Now we use 47.19 to simplify a few of these. As well, we use our usual trick of writing in index notation, reordering, and then writing as the trace in order to get a trace over these. We have:

$$iV = -e^4\int \frac{d^4\ell}{(2\pi)^4}\ell^4\left[\frac{4\ \mathrm{Tr}[\not\varepsilon_2\not\varepsilon_1\not\varepsilon_4\not\varepsilon_3]}{(\ell^2+m^2)^4} + \frac{\gamma^\mu\not\varepsilon_2\gamma^\nu\not\varepsilon_1\gamma_\mu\not\varepsilon_4\gamma_\nu\not\varepsilon_3}{(\ell^2+m^2)^2} + \frac{4\ \mathrm{Tr}[\not\varepsilon_2\not\varepsilon_1\not\varepsilon_4\not\varepsilon_3]}{(\ell^2+m^2)^4}\right] + \text{ (5 perms)}$$

Now we need to use equation 59.1.7. (*Again, it's very annoying that this derivation wasn't done in the book! Fortunately, the derivation I did in chapter 59 stands on its own*):

$$iV = -e^4\int \frac{d^4\ell}{(2\pi)^4}\ell^4\left[\frac{8\ \mathrm{Tr}[\not\varepsilon_2\not\varepsilon_1\not\varepsilon_4\not\varepsilon_3]}{(\ell^2+m^2)^4} + \frac{2\ \mathrm{Tr}[\not\varepsilon_1\gamma^\nu\not\varepsilon_2\not\varepsilon_4\gamma_\nu\not\varepsilon_3]}{(\ell^2+m^2)^2}\right] + \text{ (5 perms)}$$

Now we use equation 47.20:

$$iV = -8e^4\int \frac{d^4\ell}{(2\pi)^4}\ell^4\left[\frac{\mathrm{Tr}[\not\varepsilon_2\not\varepsilon_1\not\varepsilon_4\not\varepsilon_3]}{(\ell^2+m^2)^4} + \frac{\mathrm{Tr}[\not\varepsilon_1(\varepsilon_2\cdot\varepsilon_4)\not\varepsilon_3]}{(\ell^2+m^2)^2}\right] + \text{ (5 perms)}$$

Now we use equations 47.9 and 47.13:

$$iV = -32e^4 \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^4}{(\ell^2+m^2)^4} \left[(\varepsilon_2\cdot\varepsilon_3)(\varepsilon_1\cdot\varepsilon_4) - 2(\varepsilon_2\cdot\varepsilon_4)(\varepsilon_1\cdot\varepsilon_3) + (\varepsilon_2\cdot\varepsilon_1)(\varepsilon_4\cdot\varepsilon_3)\right] + \text{(5 perms)}$$

Now we take the other five permutations. Don't forget that there is a relative minus sign for odd permutations. Then:

$$\begin{gathered} iV = -32e^4 \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^4}{(\ell^2+m^2)^4} \left[(\varepsilon_2\cdot\varepsilon_3)(\varepsilon_1\cdot\varepsilon_4) - 2(\varepsilon_2\cdot\varepsilon_4)(\varepsilon_1\cdot\varepsilon_3) + (\varepsilon_2\cdot\varepsilon_1)(\varepsilon_4\cdot\varepsilon_3) - \right. \\ (\varepsilon_2\cdot\varepsilon_3)(\varepsilon_1\cdot\varepsilon_4) + 2(\varepsilon_2\cdot\varepsilon_1)(\varepsilon_4\cdot\varepsilon_3) - (\varepsilon_2\cdot\varepsilon_4)(\varepsilon_1\cdot\varepsilon_3) - (\varepsilon_2\cdot\varepsilon_1)(\varepsilon_3\cdot\varepsilon_4) + 2(\varepsilon_2\cdot\varepsilon_4)(\varepsilon_1\cdot\varepsilon_3) - \\ (\varepsilon_2\cdot\varepsilon_3)(\varepsilon_4\cdot\varepsilon_1) + (\varepsilon_2\cdot\varepsilon_1)(\varepsilon_3\cdot\varepsilon_4) - 2(\varepsilon_2\cdot\varepsilon_3)(\varepsilon_1\cdot\varepsilon_4) + (\varepsilon_2\cdot\varepsilon_4)(\varepsilon_3\cdot\varepsilon_1) - (\varepsilon_2\cdot\varepsilon_4)(\varepsilon_3\cdot\varepsilon_1) + \\ \left. 2(\varepsilon_2\cdot\varepsilon_3)(\varepsilon_1\cdot\varepsilon_4) - (\varepsilon_2\cdot\varepsilon_1)(\varepsilon_3\cdot\varepsilon_4) + (\varepsilon_2\cdot\varepsilon_4)(\varepsilon_3\cdot\varepsilon_1) - 2(\varepsilon_2\cdot\varepsilon_1)(\varepsilon_3\cdot\varepsilon_4) + (\varepsilon_2\cdot\varepsilon_3)(\varepsilon_1\cdot\varepsilon_4)\right] \end{gathered}$$

Simplifying:

$$V = 0$$

So there are six terms which are individually logarithmically divergent, but they sum to zero, as expected.

Let's say that didn't happen, and that some logarithmic divergence remained. Then we would have to put a term in the Lagrangian to counter this, perhaps in one of the counterterms. No external momenta are involved, so the term would have to involve something with four photon fields, ie $A^\mu A_\mu A^\nu A_\nu$. This is not gauge-invariant, so this fix (the only possible fix!) is unacceptable. It therefore follows that our QFT can hold only if the terms sum to zero, as they indeed do.