Srednicki Chapter 6

QFT Problems & Solutions

A. George

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Srednicki 6.1. (a) Find an explicit formula for $\mathcal{D}q$ in equation 6.9. Your formula should be of the form $\mathcal{D}\mathbf{q} = \mathbf{C}\prod_{\mathbf{j=1}}^{\mathbf{N}} \mathbf{dq_j}$, where C is a constant that you should compute.

We'll go back to equation 6.7:

$$\langle q'', t''|q', t'\rangle = \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-iH\delta t} \tag{6.1.1}$$

Now we let $a = q_{j+1} - q_j$. Since we're trying to get to equation 6.9, we'll make the assumptions used in that equation (see first new paragraph of page 45) now. In particular, we'll let:

$$H = Ap^2 + B_1(q)p + C(q) \tag{6.1.2}$$

$$\langle q'', t''|q', t'\rangle = \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j a} e^{-i[Ap^2+B_1(q)p+C(q)]\delta t} \tag{6.1.3}$$

Now define $B(q) = B_1(q) - \frac{a}{\delta t}$.

$$\langle q'', t''|q', t'\rangle = \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{-i[Ap^2+B(q)p+C(q)]\delta t} \tag{6.1.4}$$

Now we're ready to do the integral. We'll take the p limits at $\pm\infty$ since Srednicki tells us to use an arbitrary value, and these are good arbitrary values. Let's also define $A, B, C \rightarrow -\delta t(A, B, C)$, and supress the q-dependence for convenience. Then we have to do n integrals which look like this:

$$\frac{1}{2\pi} \int dp e^{i(Ap^2+Bp+C)} = -\frac{1}{2\pi} \frac{i^{3/2}\sqrt{\pi} e^{i(C-B^2/4A)}}{2\sqrt{A}} erfi\left(\frac{\sqrt{i}(2Ap+B)}{2\sqrt{A}}\right)\Bigg|_{-\infty}^{\infty} \tag{6.1.5}$$

This imaginary error function goes to i at infinity and $-i$ at negative infinity, provided that A is positive. Then:

$$\frac{1}{2\pi} \int dp_j e^{i(Ap^2+Bp+C)} = -\frac{1}{\sqrt{\pi}} \frac{i^{5/2} e^{i(C-B^2/4A)}}{2\sqrt{A}} \text{ for A } > 0$$

This gives:

$$\frac{1}{2\pi} \int dp_j e^{i(Ap^2+Bp+C)} = \frac{e^{i(C-B^2/4A)}}{\sqrt{4\pi i A}} \text{ for A } > 0$$

And then:

$$\frac{1}{2\pi}\int dp_j e^{i(Ap^2+Bp+C)} = \sqrt{\frac{e^{2i(C-B^2/4A)}}{4\pi i A}} \text{ for A } > 0 \tag{6.1.6}$$

Now we need to do this integral N+1 times. Hence,

$$\prod_{j=0}^{N}\frac{1}{2\pi}\int dp_j e^{i(Ap^2+Bp+C)} = \left(\frac{e^{2i(C-B^2/4A)}}{4\pi i A}\right)^{(N+1)/2}$$

Now we'll put back in the factor of $-\delta t$ in the A,B,C terms, as well as the q-dependence in the B and C terms.

$$\prod_{j=0}^{N}\frac{1}{2\pi}\int dp_j e^{i(Ap^2+B(q)p+C(q))} = \left(-\frac{e^{-2i\delta t(C(q)-B(q)^2/4A)}}{\delta t 4\pi i A}\right)^{(N+1)/2}$$

Now we put in the terms needed to recover (6.1.4).

$$\int\prod_{k=1}^{N}dq_k\prod_{j=0}^{N}\frac{1}{2\pi}dp_j e^{i(Ap^2+B(q)p+C(q))} = \int\prod_{k=1}^{N}dq_k\left(-\frac{e^{-2i\delta t(C(q)-B(q)^2/4A)}}{4\pi i A}\right)^{(N+1)/2}$$

Now we have:

$$\int\prod_{k=1}^{N}dq_k\prod_{j=0}^{N}\frac{1}{2\pi}dp_j e^{i(Ap^2+B(q)p+C(q))} = \int\prod_{k=1}^{N}dq_k\left(i\frac{e^{-2i\delta t(C(q)-B(q)^2/4A)}}{4\pi A}\right)^{(N+1)/2}$$

which is:

$$\int\prod_{k=1}^{N}dq_k\prod_{j=0}^{N}\frac{1}{2\pi}dp_j e^{i(Ap^2+B(q)p+C(q))} = \int\prod_{k=1}^{N}dq_k\left(\frac{i}{4\pi A}\right)^{(N+1)/2}\left(e^{-2i\delta t(C(q)-B(q)^2/4A)}\right)^{(N+1)/2}$$

Now we're ready to use equation (6.1.4), remembering to put $A \to -\delta t A$:

$$\langle q'', t''|q', t'\rangle = \int\prod_{k=1}^{N}dq_k\left(\frac{1}{4i\pi\delta t A}\right)^{(N+1)/2}\left(e^{-i\delta t(C(q)-B(q)^2/4A)}\right)^{N+1} \tag{6.1.7}$$

But we are being rather casual with this last term. In fact, the q's have subscripts (the js of equation 6.7) and the exponent is just a casual shorthand for a product over all the js. Let's revert to using this notation:

$$\langle q'', t''|q', t'\rangle = \int\prod_{k=1}^{N}dq_k\left(\frac{1}{4i\pi\delta t A}\right)^{(N+1)/2}\prod_{j=0}^{N}e^{-i\delta t(C(q_j)-B(q_j)^2/4A)}$$

This can obviously be rewritten as:

$$\langle q'', t''|q', t'\rangle = \int\prod_{k=1}^{N}dq_k\left(\frac{1}{4\pi i\delta t A}\right)^{(N+1)/2} e^{-i\delta t\sum_{j=0}^{N}(C(q_j)-B(q_j)^2/4A)} \tag{6.1.8}$$

Obviously this first term is the C alluded to in the problem statement.

Now let's think about the Lagrangian. The Lagrangian is given by:

$$\mathcal{L} = p\dot{q} - Ap^2 - B_1(q)p - C(q)$$

which is:

$$\mathcal{L} = p(\dot{q} - B_1(q)) - Ap^2 - C(q)$$

We use equation 6.10, which shows: $p = (\dot{q} - B_1)/(2A)$. Then,

$$\mathcal{L} = 2Ap^2 - Ap^2 - C(q)$$

which is:

$$\mathcal{L} = Ap^2 - C(q)$$

$$\mathcal{L} = A\left(\frac{\dot{q} - B_1(q)}{2A}\right)^2 - C(q)$$

Now, recall that we defined $B(q) = B_1(q) - \dot{q}$. Then,

$$\mathcal{L} = A\left(\frac{B(q)}{2A}\right)^2 - C(q)$$

$$\mathcal{L} = \frac{B(q)^2}{4A} - C(q)$$

Now we'll insert this into equation (6.1.8):

$$\langle q'', t'' | q', t' \rangle = \int \mathcal{D}q e^{i\delta t \sum_{k=1}^{N} \mathcal{L}}$$

Now let $\delta t \to 0$, which gives:

$$\langle q'', t'' | q', t' \rangle = \int \mathcal{D}q e^{i\int_{t'}^{t''} dt \mathcal{L}}$$

which is Srednicki equation 6.9. Hence, we have derived the formula, and formally defined the path integral along the way. We derived the path metric some time ago, so let me state the result again here (from equation (6.1.7)):

$$\mathcal{D}q = \left(\frac{1}{4\pi i\delta t A}\right)^{(N+1)/2} \prod_{j=1}^{N} dq_j \tag{6.1.9}$$

Note: Srednicki's solution to this problem leaves a lot to be desired. Not only does he assume a specific form for H without any justification, but he also stops at his analog of equation (6.1.8)*, without proving that* (6.1.8) *is equivalent to Srednicki equation 6.9.*

(b) In the case of a free particle, $\mathbf{V(Q) = 0}$, evaluate the path integral of equation 6.9 explicitly. Hint: Integrate over $\mathbf{q_1}$, then $\mathbf{q_2}$, etc., and look for a pattern. Express your final answer in terms of q', t', q", t", and m. Restore ħ by dimensional analysis.

We'll go back to (6.1.8):

$$\langle q'', t''|q', t'\rangle = \int \mathcal{D}q e^{-i\delta t \sum_{k=1}^{N}(C(q_k)-B(q_k)^2/4A)}$$

In this case, $A = \frac{1}{2m}$, $B_1 = 0$, $C = 0$. We can calculate $B = -\dot{q}$. Then:

$$\langle q'', t''|q', t'\rangle = \int \mathcal{D}q e^{-\frac{im}{2}\delta t \sum_{k=1}^{N} \dot{q_k}^2}$$

It's a little bit difficult to deal with the derivative, so let's write in long form.

$$\langle q'', t''|q', t'\rangle = \int \mathcal{D}q e^{-\frac{im}{2}\sum_{j=0}^{N}\frac{(q_{j+1}-q_j)^2}{\delta t}}$$

Using equation (6.1.9):

$$\langle q'', t''|q', t'\rangle = \int \left(\frac{2m}{4\pi i\delta t}\right)^{(N+1)/2} \prod_{j=1}^{N} dq_j e^{-\frac{im}{2}\sum_{j=0}^{N}\frac{(q_{j+1}-q_j)^2}{\delta t}}$$

Taking advantage of the hint, we'll do the q_1 integral:

$$\int dq_1 e^{iC(q_2-q_1)^2} e^{iC(q_1-q_0)^2}$$

where $C = -m/(2\delta t)$.

We'll expand these products:

$$\int dq_1 e^{iC\left(2q_1^2-2q_1(q_2+q_0)+(q_2^2+q_0^2)\right)}$$

This is of the form of equation (6.1.6). Now:

$$\int dq_1 e^{iC(q_2-q_1)^2} e^{iC(q_1-q_0)^2} = \sqrt{\frac{\pi}{2iC}} e^{-\frac{iC}{2}(q_2-q_0)^2}$$

which is:

$$\int dq_1 e^{iC(q_2-q_1)^2} e^{iC(q_1-q_0)^2} = \sqrt{\frac{\pi}{2iC}} e^{-\frac{iC}{2}(q_2-q_0)^2}$$

and then:

$$\int dq_1 e^{iC(q_2-q_1)^2} e^{iC(q_1-q_0)^2} = \sqrt{\frac{i\pi\delta t}{m}} e^{-\frac{iC}{2}(q_2-q_0)^2}$$

Now for the q_2 integral. I won't show all the calculus since it's the same as before, the result is:

$$\int dq_2 e^{iC(q_3-q_2)^2} e^{\frac{iC}{2}(q_2-q_0)^2} = \sqrt{\frac{4i\pi\delta t}{3m}} e^{-\frac{iC}{3}(q_3-q_0)^2}$$

At some point we see the pattern. We increase the integer in the denominator of the expononent, and weigh the factor in the coefficient accordingly. The result is:

$$\int dq_N e^{iC(q_3-q_2)^2} e^{\frac{iC}{2}(q_2-q_0)^2} = \sqrt{\frac{N2i\pi\delta t}{(N+1)m}} e^{-\frac{iC}{N+1}(q_{N+1}-q_0)^2}$$

Back to equation (6.1.9), plugging in our solutions for all the integrals:

$$\langle q'', t''|q', t'\rangle = \left(\frac{m}{2\pi i\delta t}\right)^{(N+1)/2}\left(\prod_{j=1}^{N}\sqrt{\frac{ji2\pi\delta t}{(j+1)m}}\right)e^{-\frac{iC}{j+1}(q_{j+1}-q_0)^2}$$

Note that the product acts only on the coefficient in front of the exponential, not on the exponential itself (this is because the entire q_{n-1} exponential was used to construct the q_n exponential, and so forth). As for the prefactors, everything cancels except the $N+1$ term (and the 1 term, but that is obviously unimportant). Then:

$$\langle q'', t''|q', t'\rangle = \left(\frac{m}{2\pi i\delta t}\right)^{(N+1)/2}\frac{1}{\sqrt{N+1}}\left(\frac{i2\pi\delta t}{m}\right)^{N/2}e^{-\frac{iC}{N+1}(q_{N+1}-q_0)^2}$$

which simplifies to:

$$\langle q'', t''|q', t'\rangle = \sqrt{\frac{m}{(N+1)2\pi i\delta t}}e^{\frac{im}{(N+1)2\delta t}(q_{N+1}-q_0)^2}$$

and then:

$$\langle q'', t''|q', t'\rangle = \sqrt{\frac{m}{2\pi i(t''-t')}}e^{\frac{im}{2}\frac{(q''-q')^2}{t''-t'}}$$

where we noted that $q_0 = q'$ and $q_{N+1} = q''$. Now to restore the factors of $\hbar$. The exponent needs to be dimensionless, but it currently has units of Et. So we divide by $\hbar$.

$$\langle q'', t''|q', t'\rangle = \sqrt{\frac{m}{2\pi i(t''-t')}}e^{\frac{im}{2\hbar}\frac{(q''-q')^2}{t''-t'}}$$

What units should the prefactor have? If we set $t' = t''$, then we obviously have $\langle q''|q'\rangle = \delta(q''-q')$. This will get rid of one length unit, so it has dimensions of inverse length. In the prefactor given, we should therefore divide by $\hbar$. This will give units of $(s^2)^{-1/2} = s^{-1}$, which becomes inverse length when multiplied by c (and we are not told to restore the cs. Then,

$$\langle q'', t''|q', t'\rangle = \sqrt{\frac{m}{2\pi i\hbar(t''-t')}}e^{\frac{im}{2\hbar}\frac{(q''-q')^2}{t''-t'}} \tag{6.1.10}$$

(c) Compute $\langle q'', t''|q', t'\rangle = \langle q''|e^{-iH(t''-t')}|q'\rangle$ by inserting a complete set of momentum eigenstates, and performing the integral over the momentum. Compare with your result from part (b).

$$\langle q'', t''|q', t'\rangle = \langle q''|e^{-iH(t''-t')}|q'\rangle$$

Inserting the complete set of momentum eigenstates:

$$\langle q'', t''|q', t'\rangle = \int_{-\infty}^{\infty} dp\langle q''|e^{-iH(t''-t')}|p\rangle\langle p|q'\rangle$$

The bra-ket on the right is easy to evaluate. The Hamiltonian for a free particle is given by $H = \frac{\mathbf{P}^2}{2m}$. So:

$$\langle q'', t''|q', t'\rangle = \int_{-\infty}^{\infty} dp\langle q''|e^{-i(t''-t')\frac{\mathbf{P}^2}{2m}}|p\rangle\frac{1}{\sqrt{2\pi}}e^{-ipq'}$$

The momentum operator acting on the momentum eigenstate will give the momentum eigenvalue, which we'll call p. Then,

$$\langle q'', t'' | q', t' \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp \langle q'' | e^{-i(t''-t')\frac{p^2}{2m}} | p \rangle e^{-ipq'}$$

Now the operator between the bra and the ket is just a constant, so we can take it out:

$$\langle q'', t'' | q', t' \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp e^{-i(t''-t')\frac{p^2}{2m}} \langle q'' | p \rangle e^{-ipq'}$$

which is:

$$\langle q'', t'' | q', t' \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp e^{p^2 \frac{-i(t''-t')}{2m}} e^{-ip(q'-q'')}$$

$$\langle q'', t'' | q', t' \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp e^{i\left[p^2 \frac{-(t''-t')}{2m} - p(q'-q'')\right]}$$

This is of the form of equation (6.1.6), but this time A is negative. The result is that in equation (6.1.5), we get $-2i$ rather than $2i$ from the imaginary error function. As a result, equation (6.1.6) has the i in the numerator rather than the denominator. Using this result, we have:

$$\langle q'', t'' | q', t' \rangle = \sqrt{\frac{m}{2\pi i(t''-t')}} e^{\frac{im}{2}\frac{(q''-q')^2}{t''-t'}}$$

which is the same as our result from part b, equation (6.1.10), once we add in the factors of $\hbar$ as before.