Srednicki Chapter 59

QFT Problems & Solutions

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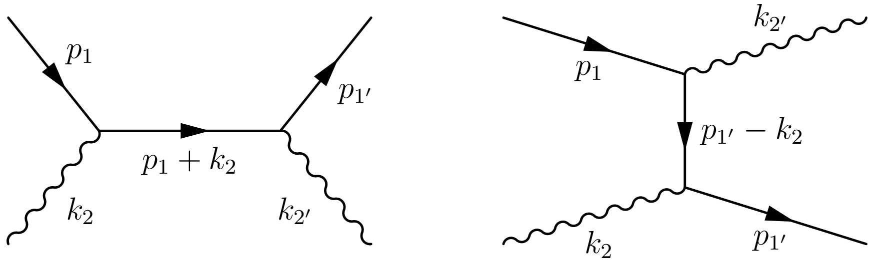
Srednicki 59.1. Compute $\langle|\mathcal{T}|^2\rangle$ for Compton scattering, $e^-\gamma \to e^-\gamma$. You should find that your result is the same as that for $e^+e^- \to \gamma\gamma$, but with $s \leftrightarrow t$, and an extra overall minus sign. This is an example of crossing symmetry; there is an overall minus sign for each fermion that is moved from the initial to the final state.

We begin by drawing the diagrams. There is an s-channel diagram and a u-channel diagram:



Note that we have neglected the arrows on the photons; their directions are obvious from conservation of momentum at the vertices. Note also that since there is only one Fermion line, the relative signs are of course positive.

Now we follow the Feynman rules, tracing the fermion lines backward, and writing all the terms we encounter. The result is:

$$i\mathcal{T} = \overline{u}_{s1'}(p_{1'})(ie\gamma_\mu)\varepsilon^{\mu}_{\lambda 2'}(k_{2'})\frac{(-i)(-\not{p}_1 - \not{k}_2 + m)}{(p_1+k_2)^2 + m^2 - i\varepsilon}(ie\gamma_\nu)\varepsilon^{\nu *}_{\lambda 2}(k_2)u_{s1}(p_1) +$$

$$\overline{u}_{s1'}(p_{1'})\varepsilon^{\mu *}_{\lambda 2}(k_2)(ie\gamma_\mu)\frac{(-i)(\not{k}_2 - \not{p}_{1'} + m)}{(k_2 - p_1')^2 + m^2 - i\varepsilon}(ie\gamma_\nu)\varepsilon^{\nu}_{\lambda 2'}(k_{2'})u_{s1}(p_1)$$

In the second term, we can switch $\mu \leftrightarrow \nu$. Further, note that these denominators resemble Mandelstam variables. Finally, $m^2 - s$ or $m^2 - u$ will vanish only if the photon has no energy and the fermion is at rest; this will not lead to scattering. As a result, these will be much larger than zero, and we can drop the infinitesimal. Putting all this together:

$$\mathcal{T} = e^2 \varepsilon^{\mu}_{\lambda 2'} \varepsilon^{\nu *}_{\lambda 2} \overline{u}_{s1'}(p_1') \left[\frac{\gamma_\mu(-\not{p}_1 - \not{k}_2 + m)\gamma_\nu}{m^2 - s} + \frac{\gamma_\nu(\not{k}_2 - \not{p}_{1'} + m)\gamma_\mu}{m^2 - u} \right] u_{s1}(p_1)$$

As per the discussion leading up to 59.6, we can easily invert this merely by swapping the indices in the brackets:

$$\overline{\mathcal{T}} = e^2 \varepsilon^{\mu *}_{\lambda 2'} \varepsilon^{\nu}_{\lambda 2} \overline{u}_{s1}(p_1) \left[\frac{\gamma_\nu(-\not{p}_1 - \not{k}_2 + m)\gamma_\mu}{m^2 - s} + \frac{\gamma_\mu(\not{k}_2 - \not{p}_{1'} + m)\gamma_\nu}{m^2 - u} \right] u_{s1'}(p_{1'})$$

Putting this together:

$$|\mathcal{T}|^2 = e^4 \varepsilon^{\mu}_{\lambda 2'} \varepsilon^{\nu *}_{\lambda 2} \varepsilon^{\sigma *}_{\lambda 2'} \varepsilon^{\rho}_{\lambda 2} \; \overline{u}_{s1'}(p_1') \left[\frac{\gamma_\mu(-\not{p}_1 - \not{k}_2 + m)\gamma_\nu}{m^2 - s} + \frac{\gamma_\nu(\not{k}_2 - \not{p}_{1'} + m)\gamma_\mu}{m^2 - u} \right] u_{s1}(p_1)$$

$$\times \; \overline{u}_{s1}(p_1) \left[\frac{\gamma_\rho(-\not{p}_1 - \not{k}_2 + m)\gamma_\sigma}{m^2 - s} + \frac{\gamma_\sigma(\not{k}_2 - \not{p}_{1'} + m)\gamma_\rho}{m^2 - u} \right] u_{s1'}(p_{1'})$$

Next we use our usual trick of writing this in index notation in order to take the trace:

$$|\mathcal{T}|^2 = e^4 \varepsilon^{\mu}_{\lambda 2'} \varepsilon^{\nu *}_{\lambda 2} \varepsilon^{\sigma *}_{\lambda 2'} \varepsilon^{\rho}_{\lambda 2} \; \mathrm{Tr} \; \left\{ u_{s1'}(p_{1'}) \overline{u}_{s1'}(p_{1'}) \left[\frac{\gamma_\mu(-\not{p}_1 - \not{k}_2 + m)\gamma_\nu}{m^2 - s} + \frac{\gamma_\nu(\not{k}_2 - \not{p}_{1'} + m)\gamma_\mu}{m^2 - u} \right] u_{s1}(p_1) \right.$$

$$\left. \times \; \overline{u}_{s1}(p_1) \left[\frac{\gamma_\rho(-\not{p}_1 - \not{k}_2 + m)\gamma_\sigma}{m^2 - s} + \frac{\gamma_\sigma(\not{k}_2 - \not{p}_{1'} + m)\gamma_\rho}{m^2 - u} \right] \right\}$$

Next we sum over the final fermion states and average over the initial fermion states:

$$\sum_{s1,s1'} |\mathcal{T}|^2 = \frac{e^4}{2} \varepsilon^{\mu}_{\lambda 2'} \varepsilon^{\nu *}_{\lambda 2} \varepsilon^{\sigma *}_{\lambda 2'} \varepsilon^{\rho}_{\lambda 2} \; \mathrm{Tr} \; \left\{ (-\not{p}_{1'} + m) \left[\frac{\gamma_\mu(-\not{p}_1 - \not{k}_2 + m)\gamma_\nu}{m^2 - s} + \frac{\gamma_\mu(\not{k}_2 - \not{p}_{1'} + m)\gamma_\mu}{m^2 - u} \right] \right.$$

$$\left. \times \; (-\not{p}_1 + m) \left[\frac{\gamma_\rho(-\not{p}_1 - \not{k}_2 + m)\gamma_\sigma}{m^2 - s} + \frac{\gamma_\sigma(\not{k}_2 - \not{p}_{1'} + m)\gamma_\rho}{m^2 - u} \right] \right\}$$

Next we average over the final photon polarization states and sum over the initial ones, with the help of 59.17:

$$\langle |\mathcal{T}|^2 \rangle = \frac{e^4}{4} g^{\mu\sigma} g^{\nu\rho} \; \mathrm{Tr} \; \left\{ (-\not{p}_{1'} + m) \left[\frac{\gamma_\mu(-\not{p}_1 - \not{k}_2 + m)\gamma_\nu}{m^2 - s} + \frac{\gamma_\nu(\not{k}_2 - \not{p}_{1'} + m)\gamma_\mu}{m^2 - u} \right] \right.$$

$$\left. \times \; (-\not{p}_1 + m) \left[\frac{\gamma_\rho(-\not{p}_1 - \not{k}_2 + m)\gamma_\sigma}{m^2 - s} + \frac{\gamma_\sigma(\not{k}_2 - \not{p}_{1'} + m)\gamma_\rho}{m^2 - u} \right] \right\}$$

Next we multiply these terms:

$$\langle|\mathcal{T}|^2\rangle = \frac{e^4}{4} g^{\mu\sigma} g^{\nu\rho} \Bigg(\text{Tr} \left\{ (-\not{p}_{1'} + m) \left[\frac{\gamma_\mu(-\not{p}_1 - \not{k}_2 + m)\gamma_\nu}{m^2 - s} \right] (-\not{p}_1 + m) \left[\frac{\gamma_\rho(-\not{p}_1 - \not{k}_2 + m)\gamma_\sigma}{m^2 - s} \right] \right\}$$

$$+ \text{Tr} \left\{ (-\not{p}_{1'} + m) \left[\frac{\gamma_\mu(-\not{p}_1 - \not{k}_2 + m)\gamma_\nu}{m^2 - s} \right] (-\not{p}_1 + m) \left[\frac{\gamma_\sigma(\not{k}_2 - \not{p}_{1'} + m)\gamma_\rho}{m^2 - u} \right] \right\}$$

$$+ \text{Tr} \left\{ (-\not{p}_{1'} + m) \left[\frac{\gamma_\nu(\not{k}_2 - \not{p}_{1'} + m)\gamma_\mu}{m^2 - u} \right] (-\not{p}_1 + m) \left[\frac{\gamma_\rho(-\not{p}_1 - \not{k}_2 + m)\gamma_\sigma}{m^2 - s} \right] \right\}$$

$$+ \text{Tr} \left\{ (-\not{p}_{1'} + m) \left[\frac{\gamma_\nu(\not{k}_2 - \not{p}_{1'} + m)\gamma_\mu}{m^2 - u} \right] (-\not{p}_1 + m) \left[\frac{\gamma_\sigma(\not{k}_2 - \not{p}_{1'} + m)\gamma_\rho}{m^2 - u} \right] \right\} \Bigg)$$

Next we use the metric.

$$\langle|\mathcal{T}|^2\rangle = \frac{e^4}{4} \Bigg\{ \frac{\text{Tr} \left[\gamma^\mu(-\not{p}_{1'} + m)\gamma_\mu(-\not{p}_1 - \not{k}_2 + m)\gamma_\nu(-\not{p}_1 + m)\gamma^\nu(-\not{p}_1 - \not{k}_2 + m) \right]}{(m^2 - s)^2}$$

$$+ \frac{\text{Tr} \left[(-\not{p}_{1'} + m)\gamma_\mu(-\not{p}_1 - \not{k}_2 + m)\gamma_\nu(-\not{p}_1 + m)\gamma^\mu(\not{k}_2 - \not{p}_{1'} + m)\gamma^\nu \right]}{(m^2 - s)(m^2 - u)}$$

$$+ \frac{\text{Tr} \left[(-\not{p}_1 + m)\gamma_\mu(-\not{p}_1 - \not{k}_2 + m)\gamma_\nu(-\not{p}_{1'} + m)\gamma^\mu(\not{k}_2 - \not{p}_{1'} + m)\gamma^\nu \right]}{(m^2 - s)(m^2 - u)}$$

$$+ \frac{\text{Tr} \left[\gamma^\mu(-\not{p}_{1'} + m)\gamma_\mu(\not{k}_2 - \not{p}_{1'} + m)\gamma_\nu(-\not{p}_1 + m)\gamma^\nu(\not{k}_2 - \not{p}_{1'} + m) \right]}{(m^2 - u)^2} \Bigg\} \tag{59.1.1}$$

Let's write this as:

$$\langle|\mathcal{T}|^2\rangle = \frac{e^4}{4} \left[\frac{\langle\Phi_{ss}\rangle}{(m^2 - s)^2} + \frac{\langle\Phi_{su}\rangle}{(m^2 - s)(m^2 - u)} + \frac{\langle\Phi_{us}\rangle}{(m^2 - s)(m^2 - u)} + \frac{\langle\Phi_{uu}\rangle}{(m^2 - u)^2} \right] \tag{59.1.2}$$

Note that by exchanging $k_2 \leftrightarrow -k_2$, and also $p_1 \leftrightarrow p_{1'}$ in (59.1.1), we exchange $\langle\Phi_{ss}\rangle \leftrightarrow \langle\Phi_{uu}\rangle$ and $\langle\Phi_{su}\rangle \leftrightarrow \langle\Phi_{us}\rangle$. We therefore need only to calculate the first two terms. Note also that these transformations are equivalent to $s \leftrightarrow u$.

Next we clean up $\langle\Phi_{ss}\rangle$ with equations 47.19 and 47.18. As for $\langle\Phi_{su}\rangle$, we will be content to simplify a bit:

$$\langle\Phi_{ss}\rangle = \text{Tr} \left[(-2\not{p}_{1'} - 4m)(-\not{p}_1 - \not{k}_2 + m)(-2\not{p}_1 - 4m)(-\not{p}_1 - \not{k}_2 + m) \right]$$

$$\langle\Phi_{su}\rangle = \text{Tr} \left[\gamma^\nu \left(\not{p}_{1'}\gamma_\mu\not{p}_1 + \not{p}_{1'}\gamma_\mu\not{k}_2 - m\not{p}_{1'}\gamma_\mu - m\gamma_\mu\not{p}_1 - m\gamma_\mu\not{k}_2 + m^2\gamma_\mu \right) \gamma_\nu(-\not{p}_1 + m)\gamma^\mu(\not{k}_2 - \not{p}_{1'} + m) \right]$$

The $\langle \Phi_{su} \rangle$ term presents some complications, as we did not do any derivations like this is chapter 47. We need to do them now. First the analog to 47.19:

$$\gamma^\mu \gamma_\nu \gamma_\mu = \gamma^\mu \left[-2g_{\nu\mu} - \gamma_\mu \gamma_\nu\right]$$

$$\gamma^\mu \gamma_\nu \gamma_\mu = -2\gamma_\nu - \gamma^\mu \gamma_\mu \gamma_\nu$$

$$\gamma^\mu \gamma_\nu \gamma_\mu = -2\gamma_\nu + 4\gamma_\nu$$

$$\gamma^\mu \gamma_\nu \gamma_\mu = 2\gamma_\nu \tag{59.1.3}$$

where the first line is equation 47.1, and the third uses equation 47.18.

Next we do the commutation for a slash and a gamma:

$$\not{a}\gamma^\mu = a_\nu \gamma^\nu \gamma^\mu$$

$$\not{a}\gamma^\mu = a_\nu \left[-2g^{\nu\mu} - \gamma^\mu \gamma^\nu\right]$$

$$\not{a}\gamma^\mu = -2a^\mu - \gamma^\mu \not{a} \tag{59.1.4}$$

Now for the analog to equation 47.20:

$$\gamma^\mu \not{a} \gamma^\nu \gamma_\mu = a_\beta \gamma^\mu \gamma^\beta \gamma^\nu \gamma_\mu$$

$$\gamma^\mu \not{a} \gamma^\nu \gamma_\mu = a_\beta \left[-2g^{\mu\beta} - \gamma^\beta \gamma^\mu\right] \gamma^\nu \gamma_\mu$$

$$\gamma^\mu \not{a} \gamma^\nu \gamma_\mu = -2\gamma^\nu \not{a} - \not{a}\gamma^\mu \gamma^\nu \gamma_\mu$$

$$\gamma^\mu \not{a} \gamma^\nu \gamma_\mu = -2\gamma^\nu \not{a} - 2\not{a}\gamma^\nu$$

$$\gamma^\mu \not{a} \gamma^\nu \gamma_\mu = 4a^\nu \tag{59.1.5}$$

where the fourth line uses equation (59.1.3) and the fifth uses equation (59.1.4). That the opposite ordering should follow the same arrangement is not obvious, but it turns out to be true, that is:

$$\gamma^\mu \gamma^\nu \not{a} \gamma_\mu = 4a^\nu \tag{59.1.6}$$

Finally, we calculate the analog to equation 47.21:

$$\gamma^\mu \not{a} \gamma^\nu \not{b} \gamma_\mu = a_\alpha b_\beta \gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta \gamma_\mu$$

$$\gamma^\mu \not{a} \gamma^\nu \not{b} \gamma_\mu = a_\alpha b_\beta \left(-2g^{\mu\alpha} - \gamma^\alpha \gamma^\mu\right) \gamma^\nu \gamma^\beta \gamma_\mu$$

$$\gamma^\mu \not{a} \gamma^\nu \not{b} \gamma_\mu = -2\gamma^\nu \not{b}\not{a} - \not{a}\gamma^\mu \gamma^\nu \not{b}\gamma_\mu$$

$$\gamma^\mu \not{a} \gamma^\nu \not{b} \gamma_\mu = -2b_\mu \gamma^\nu \gamma^\mu \not{a} - 4\not{a} b^\nu$$

$$\gamma^\mu \not{a} \gamma^\nu \not{b} \gamma_\mu = -2b_\mu \left(-2g^{\nu\mu} - \gamma^\mu \gamma^\nu\right) \not{a} - 4\not{a} b^\nu$$

$$\gamma^\mu \not{a} \gamma^\nu \not{b} \gamma_\mu = 4b^\nu \not{a} + 2\not{b}\gamma^\nu \not{a} - 4\not{a} b^\nu$$

$$\gamma^\mu \not{a} \gamma^\nu \not{b} \gamma_\mu = 2\not{b}\gamma^\nu \not{a} \tag{59.1.7}$$

where the second and fifth lines use 47.1 and the fourth line uses (59.1.6).

Now we can use these equations to simplify our expression:

$$\langle\Phi_{su}\rangle = \text{ Tr }\left[\left(\not{p}_1\gamma_\mu\not{p}_{1'} + \not{k}_2\gamma_\mu\not{p}_{1'} - 4mp_{1'\ \mu} - 4mp_{1\ \mu} - 4mk_{2\ \mu} + 2m^2\gamma_\mu\right)(-\not{p}_1 + m)\gamma^\mu(\not{k}_2 - \not{p}_{1'} + m)\right]$$

Next we break up this second term and distribute:

$$\langle\Phi_{su}\rangle = \text{ Tr }\Big[\Big(-2\not{p}_1\gamma_\mu\not{p}_{1'}\not{p}_1\gamma^\mu - 2\not{k}_2\gamma_\mu\not{p}_{1'}\not{p}_1\gamma^\mu + 4mp_{1'\ \mu}\not{p}_1\gamma^\mu + 4mp_{1\ \mu}\not{p}_1\gamma^\mu + 4mk_{2\ \mu}\not{p}_1\gamma^\mu$$

$$-2m^2\gamma_\mu\not{p}_1\gamma^\mu + 2m\not{p}_1\gamma_\mu\not{p}_{1'}\gamma^\mu + 2m\not{k}_2\gamma_\mu\not{p}_{1'}\gamma^\mu - 4m^2p_{1'\ \mu}\gamma^\mu - 4m^2p_{1\ \mu}\gamma^\mu - 4m^2k_{2\ \mu}\gamma^\mu + 2m^3\gamma_\mu\gamma^\mu\Big)$$

$$(\not{k}_2 - \not{p}_{1'} + m)\Big]$$

which is:

$$\langle\Phi_{su}\rangle = \text{ Tr }\Big[\Big(-2\not{p}_1\gamma_\mu\not{p}_{1'}\not{p}_1\gamma^\mu - 2\not{k}_2\gamma_\mu\not{p}_{1'}\not{p}_1\gamma^\mu + 4m\not{p}_{1'}\not{p}_1 + 4m\not{p}_1\not{p}_1 + 4m\not{k}_2\not{p}_1$$

$$-2m^2\gamma_\mu\not{p}_1\gamma^\mu + 2m\not{p}_1\gamma_\mu\not{p}_{1'}\gamma^\mu + 2m\not{k}_2\gamma_\mu\not{p}_{1'}\gamma^\mu - 4m^2\not{p}_{1'} - 4m^2\not{p}_1 - 4m^2\not{k}_2 + 2m^3\gamma_\mu\gamma^\mu\Big)$$

$$(\not{k}_2 - \not{p}_{1'} + m)\Big]$$

We can use up the remaining gamma matrices now:

$$\langle\Phi_{su}\rangle = \text{ Tr }\Big[\Big(-8\not{p}_1(p_{1'}\cdot p_1) - 8\not{k}_2(p_{1'}\cdot p_1) + 8m\not{p}_{1'}\not{p}_1 + 4m\not{p}_1\not{p}_1 + 4m\not{k}_2\not{p}_1 - 4m^2\not{p}_1$$

$$+4m\not{p}_1\not{p}_{1'} + 2m\not{k}_2\not{p}_{1'} - 4m^2\not{p}_{1'} - 4m^2\not{p}_1 - 4m^2\not{k}_2 - 8m^3\Big)(\not{k}_2 - \not{p}_{1'} + m)\Big]$$

This consolidates to:

$$\langle\Phi_{su}\rangle = \text{ Tr }\Big[\Big(-8\not{p}_1(p_{1'}\cdot p_1) - 8\not{k}_2(p_{1'}\cdot p_1) + 8m\not{p}_{1'}\not{p}_1 + 4m\not{p}_1\not{p}_1 + 4m\not{k}_2\not{p}_1 - 8m^2\not{p}_1$$

$$+4m\not{k}_2\not{p}_{1'} - 4m^2\not{p}_{1'} - 4m^2\not{k}_2 - 8m^3\Big)(\not{k}_2 - \not{p}_{1'} + m)\Big]$$

Now we can proceed with both terms. We multiply them out, dropping those terms with an odd number of gamma matrices. We define $\not{a} = -\not{p}_1 - \not{k}_2$ and $\not{b} = \not{k}_2 - \not{p}_{1'}$. Then:

$$\langle\Phi_{ss}\rangle = 4\text{Tr}\left[\not{p}_{1'}\not{a}\not{p}_1\not{a} + m^2\not{p}_{1'}\not{p}_1 + 4m^2\not{p}_{1'}\not{a} + 4m^2\not{a}\not{a} + 4m^2\not{p}_1\not{a} + 4m^4\right]$$

$$\langle\Phi_{su}\rangle = \text{ Tr }\Big[-8\not{p}_1\not{b}(p_{1'}\cdot p_1) - 8\not{k}_2\not{b}(p_{1'}\cdot p_1) + 8m^2\not{p}_{1'}\not{p}_1 + 4m^2\not{p}_1\not{p}_1 + 4m^2\not{k}_2\not{p}_1 - 8m^2\not{p}_1\not{b}$$

$$+4m^2\not{k}_2\not{p}_{1'} - 4m^2\not{p}_{1'}\not{b} - 4m^2\not{k}_2\not{b} - 8m^4\Big]$$

We use the traces to simplify:

$$\langle\Phi_{ss}\rangle = 16(p_{1'}\cdot a)(p_1\cdot a) - 16(p_{1'}\cdot p_1)(a\cdot a) + 16(p_{1'}\cdot a)(p_1\cdot a) - 16m^2(p_{1'}\cdot p_1)$$

$$-64m^2(p_{1'}\cdot a)-64m^2(a\cdot a)-64m^2(p_1\cdot a)+64m^4$$

$$\langle\Phi_{su}\rangle=32(p_1\cdot b)(p_{1'}\cdot p_1)+32(k_2\cdot b)(p_{1'}\cdot p_1)-32m^2(p_{1'}\cdot p_1)-16m^2(p_1\cdot p_1)-16m^2(k_2\cdot p_1)+32m^2(p_1\cdot b)$$

$$-16m^2(k_2\cdot p_{1'})+16m^2(p_{1'}\cdot b)+16m^2(k_2\cdot b)-32m^4$$

We calculate that:

$$a\cdot a=(p_1+k_2)\cdot(p_1+k_2)=p_1\cdot p_1+2p_1\cdot k_2+k_2\cdot k_2$$

$$p_1\cdot a=-p_1\cdot p_1-p_1\cdot k_2$$

$$p_{1'}\cdot a=-p_1\cdot p_{1'}-k_2\cdot p_{1'}$$

$$p_{1'}\cdot b=k_2\cdot p_{1'}-p_{1'}\cdot p_{1'}$$

$$k_2\cdot b=k_2\cdot k_2-k_2\cdot p_{1'}$$

$$p_1\cdot b=k_2\cdot p_1-p_1\cdot p_{1'}$$

Now $p_1\cdot p_1=-m^2$ and $k_2\cdot k_2=0$. Thus:

$$a\cdot a=(p_1+k_2)\cdot(p_1+k_2)=-m^2+2p_1\cdot k_2$$

$$p_1\cdot a=m^2-p_1\cdot k_2$$

$$p_{1'}\cdot a=-p_1\cdot p_{1'}-k_2\cdot p_{1'}$$

$$p_{1'}\cdot b=k_2\cdot p_{1'}+m^2$$

$$k_2\cdot b=-k_2\cdot p_{1'}$$

$$p_1\cdot b=k_2\cdot p_1-p_1\cdot p_{1'}$$

Plugging this into our expressions:

$$\langle\Phi_{ss}\rangle=16(-p_1\cdot p_{1'}-k_2\cdot p_{1'})(m^2-p_1\cdot k_2)-16(p_{1'}p_1)(-m^2+2p_1\cdot k_2)$$

$$+16(-p_1\cdot p_{1'}-k_2\cdot p_{1'})(m^2-p_1\cdot k_2)-16m^2(p_{1'}p_1)-64m^2(-p_1\cdot p_{1'}-k_2\cdot p_{1'})-64m^2(-m^2+2p_1\cdot k_2)$$

$$-64m^2(m^2-p_1\cdot k_2)+64m^4$$

$$\langle\Phi_{su}\rangle=32(k_2\cdot p_1-p_1\cdot p_{1'})(p_{1'}\cdot p_1)-32(k_2\cdot p_{1'})(p_{1'}\cdot p_1)-32m^2(p_{1'}\cdot p_1)+16m^4-16m^2(k_2\cdot p_1)$$

$$+32m^2(k_2\cdot p_1-p_1\cdot p_{1'})-16m^2(k_2\cdot p_{1'})+16m^2(k_2\cdot p_{1'}+m^2)-16m^2(k_2\cdot p_{1'})-32m^4$$

We can simplify these extensively:

$$\langle\Phi_{ss}\rangle=32\left[(k_2\cdot p_{1'})(p_1k_2)+m^2p_1\cdot p_{1'}+m^2(k_2\cdot p_{1'})-2m^2(p_1\cdot k_2)+2m^4\right]$$

$$\langle\Phi_{su}\rangle=8\left[4(p_1\cdot p_{1'})(-p_1\cdot p_{1'}-k_2\cdot p_{1'}+k_2\cdot p_1-2m^2)+2m^2(k_2\cdot p_1)-2m^2(k_2\cdot p_{1'})\right]$$

Next we need to introduce the Mandelstam variables:

$$s=-(p_1+k_2)^2=-p_1^2-k_2^2-2p_1\cdot k_2=m^2-2p_1\cdot k_2$$

$$\implies p_1\cdot k_2=\frac{m^2-s}{2}$$

$$t = -(p_1 - p_{1'})^2 = -p_1^2 - p_{1'}^2 + 2p_1 \cdot p_{1'} = 2m^2 + 2p_1 \cdot p_{1'}$$
$$\implies p_1 \cdot p_{1'} = \frac{t - 2m^2}{2}$$
$$u = -(k_2 - p_{1'})^2 = -k_2^2 - p_{1'}^2 + 2k_2 p_{1'} = m^2 + 2k_2 \cdot p_{1'}$$
$$\implies k_2 \cdot p_{1'} = \frac{u - m^2}{2}$$

Using these in our expressions:

$$\langle \Phi_{ss} \rangle = 32 \left[\left(\frac{u - m^2}{2}\right)\left(\frac{m^2 - s}{2}\right) + m^2 \left(\frac{t - 2m^2}{2}\right) + m^2 \left(\frac{u - m^2}{2}\right) - 2m^2 \left(\frac{m^2 - s}{2}\right) + 2m^4 \right]$$

$$\langle \Phi_{su} \rangle = 8 \left[4 \left(\frac{t - 2m^2}{2}\right)\left(\frac{-t + 2m^2 - u + m^2 + m^2 - s - 4m^2}{2}\right) + m^2(m^2 - s) - m^2(u - m^2) \right]$$

This simplifies extensively:

$$\langle \Phi_{ss} \rangle = 8 \left[3m^2 u - su - 3m^4 + 5m^2 s + 2m^2 t \right]$$

$$\langle \Phi_{su} \rangle = 8 \left[(t - 2m^2)(-s - t - u) + m^2(2m^2 - s - u) \right]$$

Next we recall that $s + t + u = 2m^2$. This gives:

$$\langle \Phi_{ss} \rangle = -8 \left[su - m^2(3s + u) - m^4 \right]$$

$$\langle \Phi_{su} \rangle = -8m^2 \left[t - 4m^2 \right]$$

Recalling that we can switch $s \leftrightarrow u$ to obtain the other terms, we have:

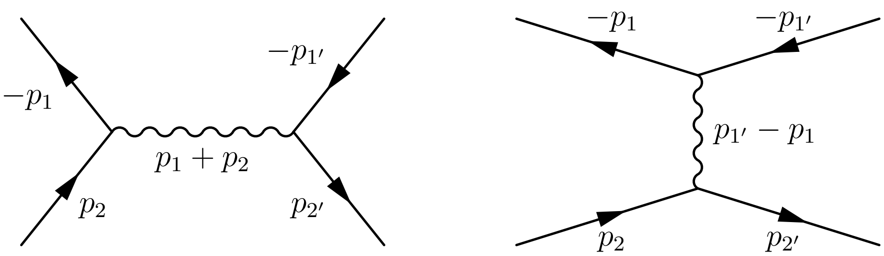
$$\langle \Phi_{us} \rangle = -8m^2 \left[t - 4m^2 \right]$$

$$\langle \Phi_{uu} \rangle = -8 \left[su - m^2(3u + s) - m^4 \right]$$

These expressions, used in equation (59.1.2), give our answer. Rather than rewriting it here, we will note that this is equal to equation 11.50, given that $\alpha = e^2/4\pi$.

Srednicki 59.2. Compute $\langle |\mathcal{T}|^2 \rangle$ for *Bhabba scattering*, $e^+e^- \to e^+e^-$.

We begin by drawing the diagrams. There is a s-channel diagram and a t-channel diagram:



Note: In Srednicki's solutions, the diagrams are drawn incorrectly. However, he eventually rectifies his mistake and arrives at the correct answer.

We use the Feynman rules to assess the values of these diagrams:

$$i\mathcal{T} = \overline{u}_{s2'}(p_{2'})(ie\gamma_\mu)v_{s1'}(p_{1'})\frac{-ig^{\mu\nu}}{(p_1+p_2)^2 - i\varepsilon}\overline{v}_{s1}(p_1)(ie\gamma_\nu)u_{s2}(p_2)$$

$$+\overline{u}_{s2'}(p_{2'})(ie\gamma_\mu)u_{s2}(p_2)\frac{-ig^{\mu\nu}}{(p_1-p_{1'})^2 - i\varepsilon}\overline{v}_{s1}(p_1)(ie\gamma_\nu)v_{s1'}(p_{1'})$$

We simplify this:

$$\mathcal{T} = -e^2\left[\frac{\overline{u}_{s2'}(p_{2'})\gamma_\mu v_{s1'}(p_{1'})\overline{v}_{s1}(p_1)\gamma^\mu u_{s2}(p_2)}{s} + \frac{\overline{u}_{s2'}(p_{2'})\gamma_\mu u_{s2}(p_2)\overline{v}_{s1}(p_1)\gamma^\mu v_{s1'}(p_{1'})}{t}\right]$$

Taking the conjugate:

$$\overline{\mathcal{T}} = -e^2\left[\frac{\overline{v}_{s1'}(p_{1'})\gamma_\mu u_{s2'}(p_{2'})\overline{u}_{s2}(p_2)\gamma^\mu v_{s1}(p_1)}{s} + \frac{\overline{u}_{s2}(p_2)\gamma_\mu u_{s2'}(p_{2'})\overline{v}_{s1'}(p_{1'})\gamma^\mu v_{s1}(p_1)}{t}\right]$$

Multiplying these, we get a big mess, which we will write as:

$$|\mathcal{T}|^2 = e^4\left[\frac{\Phi_{ss}}{s^2} + \frac{\Phi_{st}}{st} + \frac{\Phi_{ts}}{ts} + \frac{\Phi_{tt}}{t^2}\right]$$

We have, using $u_1 = u_{s1}(p_1)$ (for example) as a shorthand:

$$\Phi_{ss} = (\overline{u}_{2'}\gamma_\mu v_{1'})(\overline{v}_1\gamma^\mu u_2)(\overline{v}_{1'}\gamma_\nu u_{2'})(\overline{u}_2\gamma^\nu v_1)$$

$$\Phi_{st} = (\overline{u}_{2'}\gamma_\mu v_{1'})(\overline{v}_1\gamma^\mu u_2)(\overline{u}_2\gamma_\nu u_{2'})(\overline{v}_{1'}\gamma^\nu v_1)$$

$$\Phi_{ts} = (\overline{u}_{2'}\gamma_\mu u_2)(\overline{v}_1\gamma^\mu v_{1'})(\overline{v}_{1'}\gamma_\nu u_{2'})(\overline{u}_2\gamma^\nu v_1)$$

$$\Phi_{tt} = (\overline{u}_{2'}\gamma_\mu u_2)(\overline{v}_1\gamma^\mu v_{1'})(\overline{u}_2\gamma_\nu u_{2'})(\overline{v}_{1'}\gamma^\nu v_1)$$

Notice that everything in parenthesis is just a number. We place these in a particular order that we find pleasing, then use our usual trick to take the trace:

$$\Phi_{ss} = \text{Tr}\left(\overline{u}_{2'}\gamma_\mu v_{1'}\overline{v}_{1'}\gamma_\nu u_{2'}\right)\ \text{Tr}\left(\overline{v}_1\gamma^\mu u_2\overline{u}_2\gamma^\nu v_1\right)$$

$$\Phi_{st} = \text{Tr}\left(\overline{v}_1\gamma^\mu u_2\overline{u}_2\gamma_\nu u_{2'}\overline{u}_{2'}\gamma_\mu v_{1'}\overline{v}_{1'}\gamma^\nu v_1\right)$$

$$\Phi_{ts} = \text{Tr}\left(\overline{v}_1\gamma^\mu v_{1'}\overline{v}_{1'}\gamma_\nu u_{2'}\overline{u}_{2'}\gamma_\mu u_2\overline{u}_2\gamma^\nu v_1\right)$$

$$\Phi_{tt} = \text{Tr}\left(\overline{v}_1\gamma^\mu v_{1'}\overline{v}_{1'}\gamma^\nu v_1\right)\ \text{Tr}\left(\overline{u}_2\gamma_\nu u_{2'}\overline{u}_{2'}\gamma_\mu u_2\right)$$

Now we average over the initial states and sum over the final states. This gives:

$$\langle\Phi_{ss}\rangle = \frac{1}{4}\sum_{s1,s2,s1',s2'}\text{Tr}\left(\overline{u}_{2'}\gamma_\mu v_{1'}\overline{v}_{1'}\gamma_\nu u_{2'}\right)\ \text{Tr}\left(\overline{v}_1\gamma^\mu u_2\overline{u}_2\gamma^\nu v_1\right)$$

$$\langle\Phi_{st}\rangle = \frac{1}{4}\sum_{s1,s2,s1',s2'}\text{Tr}\left(\overline{v}_1\gamma^\mu u_2\overline{u}_2\gamma_\nu u_{2'}\overline{u}_{2'}\gamma_\mu v_{1'}\overline{v}_{1'}\gamma^\nu v_1\right)$$

$$\langle\Phi_{ts}\rangle = \frac{1}{4}\sum_{s1,s2,s1',s2'} \text{Tr}\left(\overline{v}_1\gamma^\mu v_{1'}\overline{v}_{1'}\gamma_\nu u_{2'}\overline{u}_{2'}\gamma_\mu u_2\overline{u}_2\gamma^\nu v_1\right)$$

$$\langle\Phi_{tt}\rangle = \frac{1}{4}\sum_{s1,s2,s1',s2'} \text{Tr}\left(\overline{v}_1\gamma^\mu v_{1'}\overline{v}_{1'}\gamma^\nu v_1\right)\ \text{Tr}\left(\overline{u}_2\gamma_\nu u_{2'}\overline{u}_{2'}\gamma_\mu u_2\right)$$

Now we use the completeness relations and the cyclic property of the trace:

$$\langle\Phi_{ss}\rangle = \frac{1}{4}\ \text{Tr}\Big(\gamma_\mu(-\not{p}_{1'}-m)\gamma_\nu(-\not{p}_{2'}+m)\Big)\ \text{Tr}\Big(\gamma^\mu(-\not{p}_2+m)\gamma^\nu(-\not{p}_1-m)\Big)$$

$$\langle\Phi_{st}\rangle = \frac{1}{4}\ \text{Tr}\Big(\gamma^\mu(-\not{p}_2+m)\gamma_\nu(-\not{p}_{2'}+m)\gamma_\mu(-\not{p}_{1'}-m)\gamma^\nu(-\not{p}_1-m)\Big)$$

$$\langle\Phi_{ts}\rangle = \frac{1}{4}\ \text{Tr}\Big(\gamma^\mu(-\not{p}_{1'}-m)\gamma_\nu(-\not{p}_{2'}+m)\gamma_\mu(-\not{p}_2+m)\gamma^\nu(-\not{p}_1-m)\Big)$$

$$\langle\Phi_{tt}\rangle = \frac{1}{4}\ \text{Tr}\Big(\gamma^\mu(-\not{p}_{1'}-m)\gamma^\nu(-\not{p}_1-m)\Big)\ \text{Tr}\Big(\gamma_\nu(-\not{p}_{2'}+m)\gamma_\mu(-\not{p}_2+m)\Big) \qquad (59.2.1)$$

Notice that exchanging $p_2 \leftrightarrow -p_{1'}$, equivalent to $s \leftrightarrow t$, allows us to exchange $\langle\Phi_{ss}\rangle \leftrightarrow \langle\Phi_{tt}\rangle$ and $\langle\Phi_{st}\rangle \leftrightarrow \langle\Phi_{ts}\rangle$. Therefore, we need only work out the first two of these four terms.

We distribute:

$$\langle\Phi_{ss}\rangle = \frac{1}{4}\ \text{Tr}\Big(\gamma_\mu\not{p}_{1'}\gamma_\nu\not{p}_{2'} - m^2\gamma_\mu\gamma_\nu\Big)\ \text{Tr}\Big(\gamma^\mu\not{p}_2\gamma^\nu\not{p}_1 - m^2\gamma^\mu\gamma^\nu\Big)$$

$$\begin{aligned}\langle\Phi_{st}\rangle = \frac{1}{4}\ \text{Tr}\Big(&\gamma^\mu\not{p}_2\gamma_\nu\not{p}_{2'}\gamma_\mu\not{p}_{1'}\gamma^\nu\not{p}_1 + m^2\gamma^\mu\not{p}_2\gamma_\nu\not{p}_{2'}\gamma_\mu\gamma^\nu - m^2\gamma^\mu\not{p}_2\gamma_\nu\gamma_\mu\not{p}_{1'}\gamma^\nu\\ &-m^2\gamma^\mu\not{p}_2\gamma_\nu\gamma_\mu\gamma^\nu\not{p}_1 - m^2\gamma^\mu\gamma_\nu\not{p}_{2'}\gamma_\mu\not{p}_{1'}\gamma^\nu - m^2\gamma^\mu\gamma_\nu\not{p}_{2'}\gamma_\mu\gamma^\nu\not{p}_1\\ &+m^2\gamma^\mu\gamma_\nu\gamma_\mu\not{p}_{1'}\gamma^\nu\not{p}_1 + m^4\gamma^\mu\gamma_\nu\gamma_\mu\gamma^\nu\Big)\end{aligned}$$

We use equation (59.1.7) on the first, second, fifth, and seventh terms of $\langle\Phi_{st}\rangle$. The derivation of (59.1.7) stands on its own; there is no reason to repeat it here. Then:

$$\begin{aligned}\langle\Phi_{st}\rangle = \frac{1}{4}\ \text{Tr}\Big(&2\not{p}_{2'}\gamma_\nu\not{p}_2\not{p}_{1'}\gamma^\nu\not{p}_1 + 2m^2\not{p}_{2'}\gamma_\nu\not{p}_2\gamma^\nu - m^2\gamma^\mu\not{p}_2\gamma_\nu\gamma_\mu\not{p}_{1'}\gamma^\nu\\ &-m^2\gamma^\mu\not{p}_2\gamma_\nu\gamma_\mu\gamma^\nu\not{p}_1 - 2m^2\gamma^\mu\not{p}_{1'}\gamma_\mu\not{p}_{2'} - m^2\gamma^\mu\gamma_\nu\not{p}_{2'}\gamma_\mu\gamma^\nu\not{p}_1\\ &+2m^2\gamma_\nu\not{p}_1\gamma^\nu\not{p}_{1'} + m^4\gamma^\mu\gamma_\nu\gamma_\mu\gamma^\nu\Big)\end{aligned}$$

Next we use (47.19) on the second, fifth, and seventh terms:

$$\begin{aligned}\langle\Phi_{st}\rangle = \frac{1}{4}\ \text{Tr}\Big(&2\not{p}_{2'}\gamma_\nu\not{p}_2\not{p}_{1'}\gamma^\nu\not{p}_1 + 4m^2\not{p}_{2'}\not{p}_2 - m^2\gamma^\mu\not{p}_2\gamma_\nu\gamma_\mu\not{p}_{1'}\gamma^\nu - m^2\gamma^\mu\not{p}_2\gamma_\nu\gamma_\mu\gamma^\nu\not{p}_1\\ &-4m^2\not{p}_{1'}\not{p}_{2'} - m^2\gamma^\mu\gamma_\nu\not{p}_{2'}\gamma_\mu\gamma^\nu\not{p}_1 + 4m^2\not{p}_1\not{p}_{1'} + m^4\gamma^\mu\gamma_\nu\gamma_\mu\gamma^\nu\Big)\end{aligned}$$

Next we use (47.20) on the first term:

$$\langle\Phi_{st}\rangle = \frac{1}{4}\ \text{Tr}\Big(8\not{p}_{2'}(p_2\cdot p_{1'})\not{p}_1 + 4m^2\not{p}_{2'}\not{p}_2 - m^2\gamma^\mu\not{p}_2\gamma_\nu\gamma_\mu\not{p}_{1'}\gamma^\nu - m^2\gamma^\mu\not{p}_2\gamma_\nu\gamma_\mu\gamma^\nu\not{p}_1$$

$$-4m^2\not{p}_{1'}\not{p}_{2'} - m^2\gamma^\mu\gamma_\nu\not{p}_{2'}\gamma_\mu\gamma^\nu\not{p}_1 + 4m^2\not{p}_1\not{p}_{1'} + m^4\gamma^\mu\gamma_\nu\gamma_\mu\gamma^\nu\Big)$$

Next we use (59.1.5) and (59.1.6) on the third, fourth, and sixth terms:

$$\langle\Phi_{st}\rangle = \frac{1}{4}\ \mathrm{Tr}\left(8\not{p}_{2'}(p_2\cdot p_{1'})\not{p}_1 + 4m^2\not{p}_{2'}\not{p}_2 - 4m^2 p_{2\ \nu}\not{p}_{1'}\gamma^\nu - 4m^2 p_{2\ \nu}\gamma^\nu\not{p}_1\right.$$

$$\left.-4m^2\not{p}_{1'}\not{p}_{2'} - 4m^2 p_{2'\ \nu}\gamma^\nu\not{p}_1 + 4m^2\not{p}_1\not{p}_{1'} + m^4\gamma^\mu\gamma_\nu\gamma_\mu\gamma^\nu\right)$$

Next we use (59.1.3), followed by (47.18), on the last term:

$$\langle\Phi_{st}\rangle = \frac{1}{4}\ \mathrm{Tr}\left(8\not{p}_{2'}(p_2\cdot p_{1'})\not{p}_1 + 4m^2\not{p}_{2'}\not{p}_2 - 4m^2\not{p}_{1'}\not{p}_2 - 4m^2\not{p}_2\not{p}_1\right.$$

$$\left.-4m^2\not{p}_{1'}\not{p}_{2'} - 4m^2\not{p}_{2'}\not{p}_1 + 4m^2\not{p}_1\not{p}_{1'} - 8m^4\right)$$

Finally we use (47.9) on every term except the last one, which we simply take the trace of:

$$\langle\Phi_{st}\rangle = -8(p_{2'}\cdot p_1)(p_2\cdot p_{1'}) - 4m^2(p_{2'}\cdot p_2) + 4m^2(p_{1'}\cdot p_2) + 4m^2(p_2\cdot p_1)$$

$$+4m^2(p_{1'}\cdot p_{2'}) + 4m^2(p_{2'}\cdot p_1) - 4m^2(p_1\cdot p_{1'}) - 8m^4$$

Rearranging equations 11.5 in the usual way, we have:

$$p_1\cdot p_2 = p_{1'}\cdot p_{2'} = \frac{2m^2 - s}{2}$$

$$p_1\cdot p_{1'} = p_2\cdot p_{2'} = \frac{t - 2m^2}{2}$$

$$p_1\cdot p_{2'} = p_{1'}\cdot p_2 = \frac{u - 2m^2}{2} \tag{59.2.2}$$

Using this in $\langle\Phi_{st}\rangle$:

$$\langle\Phi_{st}\rangle = -2\left(u - 2m^2\right)^2 - 2m^2(t - 2m^2) + 2m^2(u - 2m^2) + 2m^2(2m^2 - s)$$

$$+2m^2(2m^2 - s) + 2m^2(u - 2m^2) - 2m^2(t - 2m^2) - 8m^4$$

We multiply this out, finding:

$$\langle\Phi_{st}\rangle = -2u^2 - 4m^2 s - 4m^2 t + 12m^2 u - 8m^4$$

Now we have $s + t + u = 4m^2$. Using this and simplifying, we find:

$$\boxed{\langle\Phi_{st}\rangle = -2\left(u^2 - 8m^2 u + 12m^4\right)}$$

Now we turn to Φ_{ss}:

$$\langle\Phi_{ss}\rangle = \frac{1}{4}\ \mathrm{Tr}\left(\gamma_\mu\not{p}_{1'}\gamma_\nu\not{p}_{2'} - m^2\gamma_\mu\gamma_\nu\right)\ \mathrm{Tr}\left(\gamma^\mu\not{p}_2\gamma^\nu\not{p}_1 - m^2\gamma^\mu\gamma^\nu\right)$$

Distributing:

$$\langle\Phi_{ss}\rangle = \frac{1}{4}\ \mathrm{Tr}\left(\gamma_\mu \not{p}_{1'} \gamma_\nu \not{p}_{2'}\right)\ \mathrm{Tr}\left(\gamma^\mu \not{p}_2 \gamma^\nu \not{p}_1\right) - \frac{m^2}{4}\ \mathrm{Tr}\left(\gamma_\mu \not{p}_{1'} \gamma_\nu \not{p}_{2'}\right)\ \mathrm{Tr}\left(\gamma^\mu \gamma^\nu\right)$$

$$-\frac{m^2}{4}\ \mathrm{Tr}\left(\gamma_\mu \gamma_\nu\right)\ \mathrm{Tr}\left(\gamma^\mu \not{p}_2 \gamma^\nu \not{p}_1\right) + \frac{m^4}{4}\ \mathrm{Tr}\left(\gamma_\mu \gamma_\nu\right)\ \mathrm{Tr}\left(\gamma^\mu \gamma^\nu\right)$$

Next we use (47.8):

$$\langle\Phi_{ss}\rangle = \frac{1}{4}\ \mathrm{Tr}\left(\gamma_\mu \not{p}_{1'} \gamma_\nu \not{p}_{2'}\right)\ \mathrm{Tr}\left(\gamma^\mu \not{p}_2 \gamma^\nu \not{p}_1\right) + m^2\ \mathrm{Tr}\left(\gamma_\mu \not{p}_{1'} \gamma_\nu \not{p}_{2'}\right) g^{\mu\nu}$$

$$+m^2 g_{\mu\nu}\ \mathrm{Tr}\left(\gamma^\mu \not{p}_2 \gamma^\nu \not{p}_1\right) + 16m^4$$

Next we use the metric:

$$\langle\Phi_{ss}\rangle = \frac{1}{4}\ \mathrm{Tr}\left(\gamma_\mu \not{p}_{1'} \gamma_\nu \not{p}_{2'}\right)\ \mathrm{Tr}\left(\gamma^\mu \not{p}_2 \gamma^\nu \not{p}_1\right) + m^2\ \mathrm{Tr}\left(\gamma^\mu \not{p}_{1'} \gamma_\mu \not{p}_{2'}\right)$$

$$+m^2\ \mathrm{Tr}\left(\gamma_\mu \not{p}_2 \gamma^\mu \not{p}_1\right) + 16m^4$$

Next we use (47.19):

$$\langle\Phi_{ss}\rangle = \frac{1}{4}\ \mathrm{Tr}\left(\gamma_\mu \not{p}_{1'} \gamma_\nu \not{p}_{2'}\right)\ \mathrm{Tr}\left(\gamma^\mu \not{p}_2 \gamma^\nu \not{p}_1\right) + 2m^2\ \mathrm{Tr}\left(\not{p}_{1'} \not{p}_{2'}\right)$$

$$+2m^2\ \mathrm{Tr}\left(\not{p}_2 \not{p}_1\right) + 16m^4$$

Next we use (47.9):

$$\langle\Phi_{ss}\rangle = \frac{1}{4}\ \mathrm{Tr}\left(\gamma_\mu \not{p}_{1'} \gamma_\nu \not{p}_{2'}\right)\ \mathrm{Tr}\left(\gamma^\mu \not{p}_2 \gamma^\nu \not{p}_1\right) - 8m^2(p_{1'} \cdot p_{2'}) - 8m^2(p_1 \cdot p_2) + 16m^4$$

Since four-momentum is conserved, we can write this as:

$$\langle\Phi_{ss}\rangle = \frac{1}{4}\ \mathrm{Tr}\left(\gamma_\mu \not{p}_{1'} \gamma_\nu \not{p}_{2'}\right)\ \mathrm{Tr}\left(\gamma^\mu \not{p}_2 \gamma^\nu \not{p}_1\right) - 16m^2(p_1 \cdot p_2) + 16m^4$$

Now we rederive equation (47.11) with all gamma matrices; the derivation is the same, and so:

$$\mathrm{Tr}\left[\gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\rho\right] = 4g^{\mu\nu} g^{\sigma\rho} - 4g^{\mu\sigma} g^{\nu\rho} + 4g^{\mu\rho} g^{\nu\sigma}$$

Using this in our expression, we have:

$$\langle\Phi_{ss}\rangle = \left(p_{1'\ \nu} p_{2'\ \mu} - (p_{1'} \cdot p_{2'}) g_{\mu\nu} + p_{1'\ \mu} p_{2'\ \nu}\right)\ \mathrm{Tr}\left(\gamma^\mu \not{p}_2 \gamma^\nu \not{p}_1\right) - 16m^2(p_1 \cdot p_2) + 16m^4$$

Distributing:

$$\langle\Phi_{ss}\rangle = p_{1'\ \nu} p_{2'\ \mu}\ \mathrm{Tr}\left(\gamma^\mu \not{p}_2 \gamma^\nu \not{p}_1\right) - (p_{1'} \cdot p_{2'}) g_{\mu\nu}\ \mathrm{Tr}\left(\gamma^\mu \not{p}_2 \gamma^\nu \not{p}_1\right) + p_{1'\ \mu} p_{2'\ \nu}\ \mathrm{Tr}\left(\gamma^\mu \not{p}_2 \gamma^\nu \not{p}_1\right)$$

$$-16m^2(p_1 \cdot p_2) + 16m^4$$

This is:

$$\langle\Phi_{ss}\rangle = \ \mathrm{Tr}\left(\not{p}_{2'}\not{p}_2\not{p}_{1'}\not{p}_1\right) - (p_{1'}\cdot p_{2'})\ \mathrm{Tr}\left(\gamma^\mu\not{p}_2\gamma_\mu\not{p}_1\right) + \ \mathrm{Tr}\left(\not{p}_{1'}\not{p}_2\not{p}_{2'}\not{p}_1\right) - 16m^2(p_1\cdot p_2) + 16m^4$$

Using (47.19):

$$\langle\Phi_{ss}\rangle = \ \mathrm{Tr}\left(\not{p}_{2'}\not{p}_2\not{p}_{1'}\not{p}_1\right) - 2(p_{1'}\cdot p_{2'})\ \mathrm{Tr}\left(\not{p}_2\not{p}_1\right) + \ \mathrm{Tr}\left(\not{p}_{1'}\not{p}_2\not{p}_{2'}\not{p}_1\right) - 16m^2(p_1\cdot p_2) + 16m^4$$

Using (47.9):

$$\langle\Phi_{ss}\rangle = \ \mathrm{Tr}\left(\not{p}_{2'}\not{p}_2\not{p}_{1'}\not{p}_1\right) + 8(p_{1'}\cdot p_{2'})(p_1\cdot p_2) + \ \mathrm{Tr}\left(\not{p}_{1'}\not{p}_2\not{p}_{2'}\not{p}_1\right) - 16m^2(p_1\cdot p_2) + 16m^4$$

Using (47.13):

$$\langle\Phi_{ss}\rangle = 4(p_{2'}\cdot p_1)(p_2\cdot p_{1'}) - 4(p_{2'}\cdot p_{1'})(p_2\cdot p_1) + 4(p_{2'}\cdot p_2)(p_{1'}\cdot p_1) + 8(p_{1'}\cdot p_{2'})(p_1\cdot p_2) + 4(p_{1'}\cdot p_1)(p_2\cdot p_{2'})$$

$$-4(p_{1'}\cdot p_{2'})(p_2\cdot p_1) + 4(p_{1'}\cdot p_2)(p_{2'}\cdot p_1) - 16m^2(p_1\cdot p_2) + 16m^4$$

Simplifying:

$$\langle\Phi_{ss}\rangle = 8(p_{2'}\cdot p_1)(p_2\cdot p_{1'}) + 8(p_{2'}\cdot p_2)(p_{1'}\cdot p_1) - 16m^2(p_1\cdot p_2) + 16m^4$$

Using equation (59.2.2) and simplifying:

$$\boxed{\Phi_{ss} = 2\left[u^2 + t^2 - 4m^2(u+t-s) + 8m^4\right]}$$

Finally, we use $s+t+u = 4m^2$:

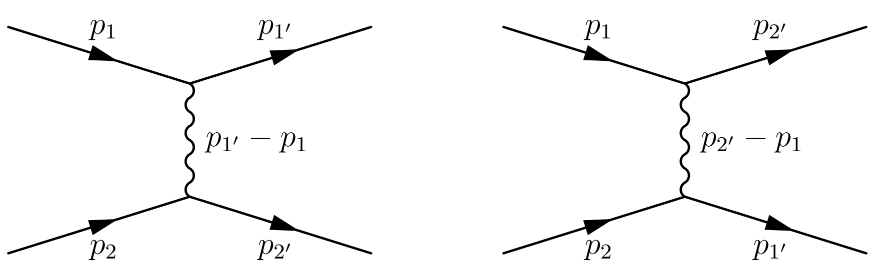
$$\boxed{\Phi_{ss} = 2\left(s^2 + 2st + 2t^2 - 8m^2 t\right)}$$

Using our boxed equations, we derive the matrix element:

$$|\mathcal{T}|^2 = 2e^4\left[\frac{s^2+2st+2t^2-8m^2t+8m^4}{s^2} - \frac{2u^2-16m^2u+24m^4}{ts} + \frac{s^2+2st+2s^2-8m^2s+8m^4}{t^2}\right]$$

Srednicki 59.3. Compute $\langle|\mathcal{T}|^2\rangle$ for Møller scattering, $e^-e^- \to e^-e^-$. You should find that your result is the same as that for $e^+e^- \to e^+e^-$, but with $s \leftrightarrow u$. This is another example of crossing symmetry.

We draw the diagrams. There is a t-channel diagram and a u-channel diagram:



We use the Feynman rules to assess the value of these diagrams:

$$i\mathcal{T} = \overline{u}_{s2'}(p_{2'})(ie\gamma_\mu)u_{s2}(p_2)\frac{-ig^{\mu\nu}}{(p_{1'}-p_1)^2 - i\varepsilon}\overline{u}_{s1'}(p_{1'})(ie\gamma_\mu)u_{s1}(p_1)$$

$$-\overline{u}_{s2'}(p_{2'})(ie\gamma_\mu)u_{s1}(p_1)\frac{-ig^{\mu\nu}}{(p_{2'}-p_1)^2 - i\varepsilon}\overline{u}_{s1'}(p_{1'})(ie\gamma_\mu)u_{s2}(p_2)$$

Simplifying:

$$\mathcal{T} = -e^2\left[\frac{\overline{u}_{s2'}(p_{2'})\gamma_\mu u_{s2}(p_2)\overline{u}_{s1'}(p_{1'})\gamma^\mu u_{s1}(p_1)}{t} - \frac{\overline{u}_{s2'}(p_{2'})\gamma_\mu u_{s1}(p_1)\overline{u}_{s1'}(p_{1'})\gamma^\mu u_{s2}(p_2)}{u}\right]$$

This gives:

$$\overline{\mathcal{T}} = -e^2\left[\frac{\overline{u}_{s2}(p_2)\gamma_\mu u_{s2'}(p_2')\overline{u}_{s1}(p_1)\gamma^\mu u_{s1'}(p_1')}{t} - \frac{\overline{u}_{s1}(p_1)\gamma_\mu u_{s2'}(p_{2'})\overline{u}_{s2}(p_2)\gamma^\mu u_{s1'}(p_{1'})}{u}\right]$$

Multiplying these, we get a big mess, which we will write as:

$$|\mathcal{T}|^2 = e^4\left[\frac{\Phi_{tt}}{t^2} + \frac{\Phi_{tu}}{tu} + \frac{\Phi_{ut}}{ut} + \frac{\Phi_{uu}}{u^2}\right]$$

We have, using $u_1 = u_{s1}(p_1)$ (for example) as a shorthand:

$$\Phi_{tt} = (\overline{u}_{2'}\gamma_\mu u_2)(\overline{u}_{1'}\gamma^\mu u_1)(\overline{u}_2\gamma_\nu u_{2'})(\overline{u}_1\gamma^\nu u_{1'})$$

$$\Phi_{tu} = -(\overline{u}_{2'}\gamma_\mu u_2)(\overline{u}_{1'}\gamma^\mu u_1)(\overline{u}_1\gamma_\nu u_{2'})(\overline{u}_2\gamma^\nu u_{1'})$$

$$\Phi_{ut} = -(\overline{u}_{2'}\gamma_\mu u_1)(\overline{u}_{1'}\gamma^\mu u_2)(\overline{u}_2\gamma_\nu u_{2'})(\overline{u}_1\gamma^\nu u_{1'})$$

$$\Phi_{uu} = (\overline{u}_{2'}\gamma_\mu u_1)(\overline{u}_{1'}\gamma^\mu u_2)(\overline{u}_1\gamma_\nu u_{2'})(\overline{u}_2\gamma^\nu u_{1'})$$

Notice that everything in parenthesis is just a number. We place these in a particular order that we find pleasing, then use our usual trick to take the trace:

$$\Phi_{tt} = \operatorname{Tr}(\overline{u}_{2'}\gamma_\mu u_2\overline{u}_2\gamma_\nu u_{2'})\ \operatorname{Tr}(\overline{u}_{1'}\gamma^\mu u_1\overline{u}_1\gamma^\nu u_{1'})$$

$$\Phi_{tu} = -\operatorname{Tr}(\overline{u}_{2'}\gamma_\mu u_2\overline{u}_2\gamma^\nu u_{1'}\overline{u}_{1'}\gamma^\mu u_1\overline{u}_1\gamma_\nu u_{2'})$$

$$\Phi_{ut} = -\operatorname{Tr}(\overline{u}_{2'}\gamma_\mu u_1\overline{u}_1\gamma^\nu u_{1'}\overline{u}_{1'}\gamma^\mu u_2\overline{u}_2\gamma_\nu u_{2'})$$

$$\Phi_{uu} = \operatorname{Tr}(\overline{u}_{2'}\gamma_\mu u_1\overline{u}_1\gamma_\nu u_{2'})\ \operatorname{Tr}(\overline{u}_{1'}\gamma^\mu u_2\overline{u}_2\gamma^\nu u_{1'})$$

Now we average over the initial states and sum over the final states. This gives:

$$\langle\Phi_{tt}\rangle = \frac{1}{4}\sum_{s1,s2,s1',s2'}\operatorname{Tr}(\overline{u}_{2'}\gamma_\mu u_2\overline{u}_2\gamma_\nu u_{2'})\ \operatorname{Tr}(\overline{u}_{1'}\gamma^\mu u_1\overline{u}_1\gamma^\nu u_{1'})$$

$$\langle\Phi_{tu}\rangle = -\frac{1}{4}\sum_{s1,s2,s1',s2'}\operatorname{Tr}(\overline{u}_{2'}\gamma_\mu u_2\overline{u}_2\gamma^\nu u_{1'}\overline{u}_{1'}\gamma^\mu u_1\overline{u}_1\gamma_\nu u_{2'})$$

$$\langle\Phi_{ut}\rangle = -\frac{1}{4}\sum_{s1,s2,s1',s2'} \mathrm{Tr}\left(\overline{u}_{2'}\gamma_\mu u_1\overline{u}_1\gamma^\nu u_{1'}\overline{u}_{1'}\gamma^\mu u_2\overline{u}_2\gamma_\nu u_{2'}\right)$$

$$\langle\Phi_{uu}\rangle = \frac{1}{4}\sum_{s1,s2,s1',s2'} \mathrm{Tr}\left(\overline{u}_{2'}\gamma_\mu u_1\overline{u}_1\gamma_\nu u_{2'}\right)\ \mathrm{Tr}\left(\overline{u}_{1'}\gamma^\mu u_2\overline{u}_2\gamma^\nu u_{1'}\right)$$

Using the completeness relations, we have:

$$\langle\Phi_{tt}\rangle = \frac{1}{4}\ \mathrm{Tr}\Big((-\not{p}_{2'}+m)\gamma_\mu(-\not{p}_2+m)\gamma_\nu\Big)\ \mathrm{Tr}\Big((-\not{p}_{1'}+m)\gamma^\mu(-\not{p}_1+m)\gamma^\nu\Big)$$

$$\langle\Phi_{tu}\rangle = -\frac{1}{4}\ \mathrm{Tr}\Big((-\not{p}_{2'}+m)\gamma_\mu(-\not{p}_2+m)\gamma^\nu(-\not{p}_{1'}+m)\gamma^\mu(-\not{p}_1+m)\gamma_\nu\Big)$$

$$\langle\Phi_{ut}\rangle = -\frac{1}{4}\ \mathrm{Tr}\Big((-\not{p}_{2'}+m)\gamma_\mu(-\not{p}_1+m)\gamma^\nu(-\not{p}_{1'}+m)\gamma^\mu(-\not{p}_2+m)\gamma_\nu\Big)$$

$$\langle\Phi_{uu}\rangle = \frac{1}{4}\ \mathrm{Tr}\Big((-\not{p}_{2'}+m)\gamma_\mu(-\not{p}_1+m)\gamma_\nu\Big)\ \mathrm{Tr}\Big((-\not{p}_{1'}+m)\gamma^\mu(-\not{p}_2+m)\gamma^\nu\Big)$$

Now we notice that this is the same as equation (59.2.1), with $p_2 \leftrightarrow -p_{2'}$, ie $s \leftrightarrow u$ (that is, $\langle\Phi_{ss}\rangle$ in the previous problem corresponds to $\langle\Phi_{uu}\rangle$ here). Thus, the answer is the same up to this change.

$$|\mathcal{T}|^2 = 2e^4\left[\frac{u^2+2tu+2t^2-8m^2t+8m^4}{u^2} - \frac{2s^2-16m^2s+24m^4}{tu} + \frac{t^2+2tu+2u^2-8m^2u+8m^4}{t^2}\right]$$