Srednicki Chapter 55

QFT Problems & Solutions

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Srednicki 55.1. Use equations 55.13-55.20 and $[A_i, A_j] = [\Pi_i, \Pi_j] = 0$ (at equal times) to verify equations 55.21-55.23.

This is our third time solving this tedious problem, so let's note from the outset that all polarization vectors $\vec{k}$ have the same magnitude, and therefore $\omega = \omega'$.

Anyway, we begin by using equation 55.16:

$$[a_\lambda(\vec{k}), a_{\lambda'}(\vec{k}')] = \left[i\varepsilon_\lambda(\vec{k}) \int d^3x e^{-ikx} \overleftrightarrow{\partial_0} \vec{A}(x), i\varepsilon_{\lambda'}(\vec{k}') \int d^3x' e^{-ik'x'} \overleftrightarrow{\partial_0} \vec{A}(x')\right]$$

This is:

$$[a_\lambda(\vec{k}), a_{\lambda'}(\vec{k}')] = -\varepsilon_\lambda(\vec{k})\varepsilon_{\lambda'}(\vec{k}') \int d^3x d^3x' \left[e^{-ikx} \overleftrightarrow{\partial_0} \vec{A}(x), e^{-ik'x'} \overleftrightarrow{\partial_0} \vec{A}(x')\right]$$

We take the derivative as indicated at the bottom of page 341. We also use equation 55.18. Then:

$$[a_\lambda(\vec{k}), a_{\lambda'}(\vec{k}')] = -\varepsilon_\lambda(\vec{k})\varepsilon_{\lambda'}(\vec{k}') \int d^3x d^3x' \left[e^{-ikx}\vec{\Pi}(x) - i\omega e^{-ikx} \vec{A}(x), e^{-ik'x'}\vec{\Pi}(x') - i\omega e^{-ik'x'} \vec{A}(x')\right]$$

Only the cross terms survive. This gives:

$$[a_\lambda(\vec{k}), a_{\lambda'}(\vec{k}')] = i\omega\varepsilon_\lambda(\vec{k})\varepsilon_{\lambda'}(\vec{k}') \int d^3x d^3x' \left\{e^{-i(kx+k'x')}[\vec{\Pi}(x), \vec{A}(x')] - e^{-i(kx+k'x')}[\vec{\Pi}(x'), \vec{A}(x)]\right\}$$

Now we take this second term and swap the two dummy variables:

$$[a_\lambda(\vec{k}), a_{\lambda'}(\vec{k}')] = i\omega\varepsilon_\lambda(\vec{k})\varepsilon_{\lambda'}(\vec{k}') \int d^3x d^3x' \left\{e^{-i(kx+k'x')} - e^{-i(kx'+k'x)}\right\} [\vec{\Pi}(x), \vec{A}(x')]$$

We now use equation 55.20. The delta function sets $x = x'$, which causes the term in braces to vanish. Thus:

$$\boxed{[a_\lambda(\vec{k}), a_{\lambda'}(\vec{k}')] = 0}$$

We can take the Hermitiian conjugate of this to find:

$$\boxed{[a^\dagger_\lambda(\vec{k}), a^\dagger_{\lambda'}(\vec{k}')] = 0}$$

The other commutator is:

$$[a_\lambda(\vec{k}), a_{\lambda'}(\vec{k}')] = -\int d^3x d^3x' \left[\vec{\varepsilon}_\lambda(\vec{k}) \cdot e^{-ikx}\overleftrightarrow{\partial}_0 \vec{A}(x), \vec{\varepsilon}^*_{\lambda'}(\vec{k}') \cdot e^{ik'x'}\overleftrightarrow{\partial}_0 \vec{A}(x')\right]$$

We write this dot product in index notation:

$$[a_\lambda(\vec{k}), a_{\lambda'}(\vec{k}')] = -\varepsilon^i_\lambda(\vec{k})\varepsilon^{*j}_{\lambda'}(\vec{k}') \int d^3x d^3x' \left[e^{-ikx}\overleftrightarrow{\partial}_0 A^i(x), e^{ik'x'}\overleftrightarrow{\partial}_0 A^j(x')\right]$$

Now we take the derivative, and recall that $\dot{A} = -\Pi$. Then:

$$[a_\lambda(\vec{k}), a_{\lambda'}(\vec{k}')] = -\varepsilon^i_\lambda(\vec{k})\varepsilon^{*j}_{\lambda'}(\vec{k}') \int d^3x d^3x' \left[e^{-ikx}\Pi^i(x) + i\omega e^{-ikx}A^i(x), e^{ik'x'}\Pi^j(x') - i\omega e^{ik'x'}A^j(x')\right]$$

Next we recall $[A^i(x), A^j(x')] = -[\Pi^i(x), \Pi^j(x')] = -0$.

$$[a_\lambda(\vec{k}), a_{\lambda'}(\vec{k}')] = -i\omega\varepsilon^i_\lambda(\vec{k})\varepsilon^{*j}_{\lambda'}(\vec{k}') \int d^3x\ d^3x'\ e^{i(k'x'-kx)} \left\{-\left[\Pi^i(x), A^j(x')\right] + \left[A^i(x), \Pi^j(x')\right]\right\}$$

Now we have:

$$[a_\lambda(\vec{k}), a_{\lambda'}(\vec{k}')] = -i\omega\varepsilon^i_\lambda(\vec{k})\varepsilon^{*j}_{\lambda'}(\vec{k}') \int d^3x\ d^3x'\ e^{i(k'x'-kx)} \left\{\left[A^j(x'), \Pi^i(x)\right] + \left[A^i(x), \Pi^j(x')\right]\right\}$$

Now we use equation 55.20:

$$[a_\lambda(\vec{k}), a_{\lambda'}(\vec{k}')] = \omega\varepsilon^i_\lambda(\vec{k})\varepsilon^{*j}_{\lambda'}(\vec{k}') \int d^3x\ d^3x'\ \frac{d^3k''}{(2\pi)^3}\ e^{i(k'x'-kx)} \left\{e^{i\vec{k}''\cdot(\vec{x}'-\vec{x})}\left(\delta_{ji} - \frac{k''_j k''_i}{k''^2}\right)\right.$$
$$\left. + e^{i\vec{k}''\cdot(\vec{x}-\vec{x}')}\left(\delta_{ij} - \frac{k''_i k''_j}{k''^2}\right)\right\}$$

Now we can change the integral $\vec{k}'' \to -\vec{k}''$. Thus:

$$[a_\lambda(\vec{k}), a_{\lambda'}(\vec{k}')] = 2\omega\varepsilon^i_\lambda(\vec{k})\varepsilon^{*j}_{\lambda'}(\vec{k}') \int d^3x\ d^3x'\ \frac{d^3k''}{(2\pi)^3}\ e^{i(k'x'-kx)} \left\{e^{i\vec{k}''\cdot(\vec{x}-\vec{x}')}\left(\delta_{ij} - \frac{k''_i k''_j}{k''^2}\right)\right\}$$

Now in the second term, one of the dot products is $\vec{\varepsilon}_\lambda(\vec{k}) \cdot \vec{k}''$, where $\vec{k}''$ is integrated over all possible values. The result, therefore, is zero: for every component of $\vec{k}''$ in the same direction as $\vec{\varepsilon}_\lambda(\vec{k})$, the $-\vec{k}''$ will have the component in the opposite direction, resulting in a cancellation. Thus, the second term vanishes.

Note that this has nothing to do with equation 55.13, as Srednicki claims in his solutions. Vectors $\vec{k}$, $\vec{k}'$, and $\vec{k}''$ are all distinct.

This gives:

$$[a_\lambda(\vec{k}), a_{\lambda'}(\vec{k}')] = 2\omega\varepsilon^i_\lambda(\vec{k})\varepsilon^{*j}_{\lambda'}(\vec{k}') \int d^3x\ d^3x'\ \frac{d^3k''}{(2\pi)^3}\ e^{i(k'x'-kx)}e^{i\vec{k}''\cdot(\vec{x}-\vec{x}')}\delta_{ij}$$

This dot product gives:

$$[a_\lambda(\vec{k}), a_{\lambda'}(\vec{k'})] = 2\omega \varepsilon_\lambda(\vec{k}) \cdot \varepsilon^*_{\lambda'}(\vec{k'}) \int d^3x \; d^3x' \; \frac{d^3k''}{(2\pi)^3} \; e^{i(k'x'-kx)} e^{i\vec{k}''\cdot(\vec{x}-\vec{x}')}$$

Now we can use 3.27 and 55.14:

$$[a_\lambda(\vec{k}), a_{\lambda'}(\vec{k'})] = 2\omega \delta_{\lambda\lambda'} \int d^3x \; d^3x' \; e^{i(k'x'-kx)} \delta^3(\vec{x}-\vec{x}')$$

Doing the x' integral:

$$[a_\lambda(\vec{k}), a_{\lambda'}(\vec{k'})] = 2\omega \delta^3_{\lambda\lambda'} \int d^3x \; e^{i(k'-k)x}$$

The temporal components of the exponential vanish, since $\omega' = \omega$ as discussed previously. Now we use 3.27 again on the spatial parts, which gives:

$$\boxed{[a_\lambda(\vec{k}), a_{\lambda'}(\vec{k'})] = 2\omega (2\pi)^3 \delta_{\lambda\lambda'} \delta^3(k'-k)}$$

Srednicki 55.2. Use equations 55.11, 55.14, 55.19, and 55.21-55.23 to verify equation 55.24.

We begin with equation 55.19:

$$\mathcal{H} = \frac{1}{2}\Pi_i\Pi_i + \frac{1}{2}\nabla_j A_i \nabla_j A_i - J_i A_i + H_{coul}$$

Now we have equation 55.11, and recall that $\Pi = \dot{A}$. Then:

$$\mathcal{H} = \frac{(i\omega)^2}{2} \sum_{\lambda=\pm}\sum_{\lambda'=\pm} \int \widetilde{dk}\widetilde{dk'} \left[-\varepsilon^{*i}_\lambda(\vec{k}) a_\lambda(\vec{k}) e^{ikx} + \varepsilon^i_\lambda(\vec{k}) a^\dagger_\lambda(\vec{k}) e^{-ikx}(i\omega) \right] \left[-\varepsilon^{*i}_\lambda(\vec{k'}) a_\lambda(\vec{k'}) e^{ik'x} \right.$$

$$\left. + \varepsilon^i_\lambda(\vec{k'}) a^\dagger_\lambda(\vec{k'}) e^{-ik'x}(i\omega) \right] + \frac{(i)^2}{2} \sum_{\lambda=\pm}\sum_{\lambda'=\pm} \int \widetilde{dk}\widetilde{dk'} \left[\varepsilon^{*i}_\lambda(\vec{k}) a_\lambda(\vec{k})(k_j) e^{ikx} + \varepsilon^i_\lambda(\vec{k}) a^\dagger_\lambda(\vec{k})(-k_j) e^{-ikx} \right]$$

$$+ \left[\varepsilon^{*i}_\lambda(\vec{k'}) a_\lambda(\vec{k'})(k'_j) e^{ik'x} + \varepsilon^i_\lambda(\vec{k'}) a^\dagger_\lambda(\vec{k'})(-k'_j) e^{-ik'x} \right] - \vec{J}(x)\cdot\vec{A}(x) + \mathcal{H}_{\text{coul}}$$

Now we do some multiplication and factor terms from the first line with terms from the second and third lines. The result is:

$$\mathcal{H} = -\frac{1}{2} \sum_{\lambda=\pm}\sum_{\lambda'=\pm} \int \widetilde{dk}\widetilde{dk'} \left\{ \vec{\varepsilon}^*_\lambda(\vec{k}) \cdot \vec{\varepsilon}^*_{\lambda'}(\vec{k'}) a_\lambda(\vec{k}) a_\lambda(\vec{k'}) e^{i(k+k')x} (\omega^2 + \vec{k}\cdot\vec{k'}) \right.$$

$$+ \vec{\varepsilon}_\lambda(\vec{k}) \cdot \vec{\varepsilon}_{\lambda'}(\vec{k'}) a^\dagger_\lambda(\vec{k}) a^\dagger_{\lambda'}(\vec{k'}) e^{-i(k+k')x} (\omega^2 + \vec{k}\cdot\vec{k'}) + \vec{\varepsilon}_\lambda(\vec{k}) \cdot \vec{\varepsilon}^*_{\lambda'}(\vec{k'}) a^\dagger_\lambda(\vec{k}) a_{\lambda'}(\vec{k'}) e^{i(k'-k)x} (-\vec{k}\cdot\vec{k'} - \omega^2)$$

$$\left. + \vec{\varepsilon}^*_\lambda(\vec{k}) \cdot \vec{\varepsilon}_{\lambda'}(\vec{k'}) a_\lambda(\vec{k}) a^\dagger_{\lambda'}(\vec{k'}) e^{i(k-k')x} (-\vec{k}\cdot\vec{k'} - \omega^2) \right\} - \vec{J}(x)\cdot\vec{A}(x) + \mathcal{H}_{\text{coul}}$$

The next step is to move from the Hamiltonian density to the Hamiltonian, by integrating both sides.

$$H = -\frac{1}{2} \sum_{\lambda=\pm}\sum_{\lambda'=\pm} \int \widetilde{dk}\widetilde{dk'} d^3x \left\{ \vec{\varepsilon}^*_\lambda(\vec{k}) \cdot \vec{\varepsilon}^*_{\lambda'}(\vec{k'}) a_\lambda(\vec{k}) a_\lambda(\vec{k'}) e^{i(k+k')x} (\omega^2 + \vec{k}\cdot\vec{k'}) \right.$$

$$+\vec{\varepsilon}_\lambda(\vec{k})\cdot\vec{\varepsilon}_{\lambda'}(\vec{k}')a^\dagger_\lambda(\vec{k})a^\dagger_{\lambda'}(\vec{k}')e^{-i(k+k')x}(\omega^2+\vec{k}\cdot\vec{k}')-\vec{\varepsilon}_\lambda(\vec{k})\cdot\vec{\varepsilon}^*_{\lambda'}(\vec{k}')a^\dagger_\lambda(\vec{k})a_{\lambda'}(\vec{k}')e^{i(k'-k)x}(\vec{k}\cdot\vec{k}'+\omega^2)$$

$$-\vec{\varepsilon}^*_\lambda(\vec{k})\cdot\vec{\varepsilon}_{\lambda'}(\vec{k}')a_\lambda(\vec{k})a^\dagger_{\lambda'}(\vec{k}')e^{i(k-k')x}(\vec{k}\cdot\vec{k}'+\omega^2)\Big\}-\int d^3x\ \vec{J}(x)\cdot\vec{A}(x)+H_{\rm coul}$$

Now the last two terms look good! In the first term, we use 3.27 to do the x-integral:

$$H=-\frac{(2\pi)^3}{2}\sum_{\lambda=\pm}\sum_{\lambda'=\pm}\int\widetilde{dk}\ \widetilde{dk'}\Big\{\vec{\varepsilon}^*_\lambda(\vec{k})\cdot\vec{\varepsilon}^*_{\lambda'}(\vec{k}')a_\lambda(\vec{k})a_\lambda(\vec{k}')\delta^3(\vec{k}+\vec{k}')e^{-2i\omega t}(\omega^2+\vec{k}\cdot\vec{k}')$$

$$+\vec{\varepsilon}_\lambda(\vec{k})\cdot\vec{\varepsilon}_{\lambda'}(\vec{k}')a^\dagger_\lambda(\vec{k})a^\dagger_{\lambda'}(\vec{k}')\delta^3(\vec{k}+\vec{k}')e^{2i\omega t}(\omega^2+\vec{k}\cdot\vec{k}')-\vec{\varepsilon}_\lambda(\vec{k})\cdot\vec{\varepsilon}^*_{\lambda'}(\vec{k}')a^\dagger_\lambda(\vec{k})a_{\lambda'}(\vec{k}')\delta^3(\vec{k}'-\vec{k})(\vec{k}\cdot\vec{k}'+\omega^2)$$

$$-\vec{\varepsilon}^*_\lambda(\vec{k})\cdot\vec{\varepsilon}_{\lambda'}(\vec{k}')a_\lambda(\vec{k})a^\dagger_{\lambda'}(\vec{k}')\delta^3(\vec{k}-\vec{k}')(\vec{k}\cdot\vec{k}'+\omega^2)\Big\}-\int d^3x\ \vec{J}(x)\cdot\vec{A}(x)+H_{\rm coul}$$

Now we can use the delta functions to perform the k' integral:

$$H=-\frac{(2\pi)^3}{2}\sum_{\lambda=\pm}\sum_{\lambda'=\pm}\int\widetilde{dk}\ \frac{1}{(2\pi)^3 2\omega}\Big\{\vec{\varepsilon}^*_\lambda(\vec{k})\cdot\vec{\varepsilon}^*_{\lambda'}(-\vec{k})a_\lambda(\vec{k})a_\lambda(-\vec{k})e^{-2i\omega t}(\omega^2-|\vec{k}|^2)$$

$$+\vec{\varepsilon}_\lambda(\vec{k})\cdot\vec{\varepsilon}_{\lambda'}(-\vec{k})a^\dagger_\lambda(\vec{k})a^\dagger_{\lambda'}(-\vec{k})e^{2i\omega t}(\omega^2-|\vec{k}|^2)-\vec{\varepsilon}_\lambda(\vec{k})\cdot\vec{\varepsilon}^*_{\lambda'}(\vec{k})a^\dagger_\lambda(\vec{k})a_{\lambda'}(\vec{k})(|\vec{k}|^2+\omega^2)$$

$$-\vec{\varepsilon}^*_\lambda(\vec{k})\cdot\vec{\varepsilon}_{\lambda'}(\vec{k})a_\lambda(\vec{k})a^\dagger_{\lambda'}(\vec{k})(|\vec{k}|^2+\omega^2)\Big\}-\int d^3x\ \vec{J}(x)\cdot\vec{A}(x)+H_{\rm coul}$$

Now we recall $\omega^2=|\vec{k}|^2$, so the first two terms vanish, and we can third and fourth terms:

$$H=\frac{(2\pi)^3}{2}\sum_{\lambda=\pm}\sum_{\lambda'=\pm}\int\widetilde{dk}\ \frac{1}{(2\pi)^3}\Big[\vec{\varepsilon}_\lambda(\vec{k})\cdot\vec{\varepsilon}^*_{\lambda'}(\vec{k})a^\dagger_\lambda(\vec{k})a_{\lambda'}(\vec{k})\omega+\vec{\varepsilon}^*_\lambda(\vec{k})\cdot\vec{\varepsilon}_{\lambda'}(\vec{k})a_\lambda(\vec{k})a^\dagger_{\lambda'}(\vec{k})\omega\Big]$$

$$-\int d^3x\ \vec{J}(x)\cdot\vec{A}(x)+H_{\rm coul}$$

Now we use equation 55.14:

$$H=\frac{(2\pi)^3}{2}\sum_{\lambda=\pm}\sum_{\lambda'=\pm}\int\widetilde{dk}\ \frac{1}{(2\pi)^3}\Big[\delta_{\lambda\lambda'}a^\dagger_\lambda(\vec{k})a_{\lambda'}(\vec{k})\omega+\delta_{\lambda\lambda'}a_\lambda(\vec{k})a^\dagger_{\lambda'}(\vec{k})\omega\Big]$$

$$-\int d^3x\ \vec{J}(x)\cdot\vec{A}(x)+H_{\rm coul}$$

We can do the sum over λ':

$$H=\frac{1}{2}\sum_{\lambda=\pm}\int\widetilde{dk}\ \omega\Big[a^\dagger_\lambda(\vec{k})a_\lambda(\vec{k})+a_\lambda(\vec{k})a^\dagger_\lambda(\vec{k})\Big]-\int d^3x\ \vec{J}(x)\cdot\vec{A}(x)+H_{\rm coul}$$

We combine these first two terms with 55.23. Then:

$$H=\frac{1}{2}\sum_{\lambda=\pm}\int\widetilde{dk}\ \omega\Big[2a^\dagger_\lambda(\vec{k})a_\lambda(\vec{k})+(2\pi)^3 2\omega\delta^3(0)\Big]-\int d^3x\ \vec{J}(x)\cdot\vec{A}(x)+H_{\rm coul}$$

which is:

$$H = \sum_{\lambda=\pm} \int \widetilde{dk}\ \omega\ a_\lambda^\dagger(\vec{k}) a_\lambda(\vec{k}) + \sum_{\lambda=\pm} \int \widetilde{dk}\ (2\pi)^3 \omega^2 \delta^3(0) - \int d^3x\ \vec{J}(x) \cdot \vec{A}(x) + H_{\text{coul}}$$

Using 3.18:

$$H = \sum_{\lambda=\pm} \int \widetilde{dk}\ \omega\ a_\lambda^\dagger(\vec{k}) a_\lambda(\vec{k}) + \sum_{\lambda=\pm} \int \frac{d^3k}{(2\pi)^3 2\omega}\ (2\pi)^3 \omega^2 \delta^3(0) - \int d^3x\ \vec{J}(x) \cdot \vec{A}(x) + H_{\text{coul}}$$

We simplify, and take the sum:

$$H = \sum_{\lambda=\pm} \int \widetilde{dk}\ \omega\ a_\lambda^\dagger(\vec{k}) a_\lambda(\vec{k}) + \int d^3k\ \omega \delta^3(0) - \int d^3x\ \vec{J}(x) \cdot \vec{A}(x) + H_{\text{coul}}$$

This gives:

$$H = \sum_{\lambda=\pm} \int \widetilde{dk}\ \omega\ a_\lambda^\dagger(\vec{k}) a_\lambda(\vec{k}) + 2\mathcal{E}_0 (2\pi)^3 \delta^3(0) - \int d^3x\ \vec{J}(x) \cdot \vec{A}(x) + H_{\text{coul}}$$

As in chapter 3, we interpret $V = (2\pi)^3 \delta^3(0)$:

$$H = \sum_{\lambda=\pm} \int \widetilde{dk}\ \omega\ a_\lambda^\dagger(\vec{k}) a_\lambda(\vec{k}) + 2\mathcal{E}_0 V - \int d^3x\ \vec{J}(x) \cdot \vec{A}(x) + H_{\text{coul}}$$

which is equation 55.24.