Srednicki Chapter 52

QFT Problems & Solutions

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July 27, 2013

Srednicki 52.1. Compute the one-loop contributions to the anomalous dimensions of m, M, Ψ, and ϕ.

We begin by writing the renormalized Lagrangian:

$$\mathcal{L} = iZ_\Psi \overline{\Psi} \partial\!\!\!/ \Psi - Z_m m \overline{\Psi}\Psi - \frac{1}{2} Z_\phi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} Z_M M^2 \phi^2 + iZ_g g \phi \overline{\Psi} \gamma_5 \Psi - \frac{1}{24} Z_\lambda \lambda \phi^4$$

and the bare Lagrangian:

$$\mathcal{L}_0 = i\overline{\Psi}_0 \partial\!\!\!/ \Psi_0 - m_0 \overline{\Psi}_0 \Psi_0 - \frac{1}{2} \partial^\mu \phi_0 \partial_\mu \phi_0 - \frac{1}{2} M_0^2 \phi_0^2 + ig_0 \phi_0 \overline{\Psi}_0 \gamma_5 \Psi_0 - \frac{1}{24} \lambda_0 \phi_0^4$$

We equate these Lagrangians term-by-term, beginning with the third and first terms. We therefore learn:

$$\phi_0 = Z_\phi^{1/2} \phi$$

$$\Psi_0 = Z_\Psi^{1/2} \Psi$$

$$m_0 = Z_\Psi^{-1} m Z_m$$

$$M_0 = Z_\phi^{-1/2} Z_M^{1/2} M$$

$$g_0 = Z_\Psi^{-1} Z_\phi^{-1/2} Z_g g$$

$$\lambda_0 = Z_\phi^{-2} \lambda Z_\lambda$$

Let's start with the anomalous dimension of m. Then:

$$m_0 = Z_\Psi^{-1} m Z_m$$

Taking the natural logarithm of both side:

$$\log m_0 = \log(Z_\Psi^{-1} Z_m) + \log m$$

The bare fields must be independent of μ, as discusse in chapter 28. Thus,

$$0 = \frac{d}{d\log\mu} \log m_0 = \frac{d}{d\log\mu} \log(Z_\Psi^{-1} Z_m) + \frac{d}{d\log\mu} \log m$$

which is:

$$\frac{1}{m}\frac{dm}{d\log\mu} = -\frac{d}{d\log\mu}\log(Z_\Psi^{-1}Z_m)$$

This left hand side is defined to be the anomalous dimension. Let us further note from chapter 51 that:

$$Z_\Psi = 1 - \frac{g^2}{16\pi^2\varepsilon}$$

$$Z_m = 1 - \frac{g^2}{8\pi^2\varepsilon}$$

Thus:

$$\gamma_m = -\frac{d}{d\log\mu}\log\left[\left(1-\frac{g^2}{16\pi^2\varepsilon}\right)^{-1}\left(1-\frac{g^2}{8\pi^2\varepsilon}\right)\right]$$

Now use the chain rule:

$$\gamma_m = -\frac{dg}{d\log\mu}\frac{d}{dg}\log\left[\left(1-\frac{g^2}{16\pi^2\varepsilon}\right)^{-1}\left(1-\frac{g^2}{8\pi^2\varepsilon}\right)\right]$$

Now we want to expand these logarithms, but it's a bit unclear where to expand it from. The ε in the denominator makes the second term in each binomial blow up – but don't forget that there is also a factor of g^2, which has the opposite effect. Thus, the second term is the small one, and we expand around $g^2/\varepsilon = 0$. Thus:

$$\gamma_m = -\frac{dg}{d\log\mu}\frac{d}{dg}\log\left[\left(1+\frac{g^2}{16\pi^2\varepsilon}\right)\left(1-\frac{g^2}{8\pi^2\varepsilon}\right)\right]$$

Now we use equation 52.11, and multiply:

$$\gamma_m = \frac{1}{2}\varepsilon g\frac{d}{dg}\log\left[\left(1+\frac{g^2}{16\pi^2\varepsilon}\right)\left(1-\frac{g^2}{8\pi^2\varepsilon}\right)\right]$$

This gives:

$$\gamma_m = \frac{1}{2}\varepsilon g\frac{d}{dg}\log\left[1-\frac{g^2}{16\pi^2\varepsilon}+O\left(\frac{g^4}{\varepsilon^2}\right)\right]$$

and so:

$$\gamma_m = -\frac{\varepsilon g}{2}\frac{d}{dg}\frac{g^2}{16\pi^2\varepsilon}$$

$$\boxed{\gamma_m = -\frac{g^2}{16\pi^2}}$$

Note: This disagrees by a minus sign from Srednicki's solution. I hate disagreeing with Srednicki, but I came across an independent solution by Andre Schneider from Indiana University; his answer agrees with mine. Further, the other parts of this problem do agree with Srednicki's solution.

Next we have:

$$M_0 = Z_\phi^{-1/2}Z_M^{1/2}M$$

By the same logic, we obtain:

$$\gamma_M = -\frac{1}{2}\frac{d}{d\log\mu}\log(Z_\phi^{-1}Z_M)$$

which is:

$$\gamma_M = -\frac{1}{2}\frac{d}{d\log\mu}\log\left[\left(1+\frac{\lambda}{16\pi^2\varepsilon}-\frac{g^2m^2}{2\pi^2M^2\varepsilon}\right)\left(1+\frac{g^2}{4\pi^2\varepsilon}\right)\right]$$

Now we expand the logarithm:

$$\gamma_M = -\frac{1}{2}\frac{d}{d\log\mu}\left[\frac{\lambda}{16\pi^2\varepsilon}-\frac{g^2m^2}{2\pi^2M^2\varepsilon}+\frac{g^2}{4\pi^2\varepsilon}\right]$$

and use the chain rule:

$$\gamma_M = -\frac{1}{2}\left(\frac{d}{dg}\frac{dg}{d\log\mu}+\frac{d}{d\lambda}\frac{d\lambda}{d\log\mu}\right)\left[\frac{\lambda}{16\pi^2\varepsilon}-\frac{g^2m^2}{2\pi^2M^2\varepsilon}+\frac{g^2}{4\pi^2\varepsilon}\right]$$

Using equation 52.11 and 52.12:

$$\gamma_M = -\frac{1}{2}\left(-\frac{1}{2}\varepsilon g\frac{d}{dg}-\varepsilon\lambda\frac{d}{d\lambda}\right)\left[\frac{\lambda}{16\pi^2\varepsilon}-\frac{g^2m^2}{2\pi^2M^2\varepsilon}+\frac{g^2}{4\pi^2\varepsilon}\right]$$

This is:

$$\gamma_M = -\frac{1}{2}\left[(-\varepsilon\lambda)\frac{1}{16\pi^2\varepsilon}-(-\frac{1}{2}\varepsilon g)\frac{2gm^2}{2\pi^2M^2\varepsilon}+(-\frac{1}{2}\varepsilon g)\frac{2g}{4\pi^2\varepsilon}\right]$$

Finally:

$$\boxed{\gamma_M = \frac{\lambda}{32\pi^2}-\frac{g^2m^2}{4\pi^2M^2}+\frac{g^2}{8\pi^2}}$$

Now for the anomalous dimension of the fields. Recall that these are defined differently; using equation 28.36.

$$\gamma_\phi = \frac{1}{2}\frac{d\ln Z_\phi}{d\ln\mu}$$

Using the chain rule, expanding the logarithm, and simplifing, as before, we have:

$$\gamma_\phi = \frac{1}{2}\frac{d}{dg}\frac{dg}{d\ln\mu}\ln Z_\phi$$

$$\gamma_\phi = -\frac{1}{4}\varepsilon g\frac{d}{dg}\left(-\frac{g^2}{4\pi^2\varepsilon}\right)$$

$$\boxed{\gamma_\phi = \frac{g^2}{8\pi^2}}$$

Similarly,

$$\gamma_\Psi = \frac{1}{2}\left(-\frac{1}{2}\varepsilon g\right)\frac{d}{dg}\log\left(1-\frac{g^2}{16\pi^2\varepsilon}\right)$$

$$\boxed{\gamma_\Psi = \frac{g^2}{32\pi^2}}$$

Srednicki 52.2. Consider the theory of problem 51.3. Compute the one-loop contributions to the beta function for g, λ, and κ, and to the anomalous dimension of m, M, Ψ, and ϕ.

All we've changed from the chapter is the interaction Lagrangian:

$$\mathcal{L}_1 = Z_g g\phi\overline{\Psi}\Psi + \frac{1}{6}Z_\kappa\kappa\phi^3 + \frac{1}{24}Z_\lambda\lambda\phi^4$$

The other terms of the Lagrangian are the same. Further, the difference in the Dirac term does not affect the bare field-renormalized field relationships that we found in the previous problem. Thus, we can quote from the previous problem:

$$\phi_0 = Z_\phi^{1/2}\phi$$

$$\Psi_0 = Z_\Psi^{1/2}\Psi$$

$$m_0 = Z_\Psi^{-1}mZ_m$$

$$M_0 = Z_\phi^{-1/2}Z_M^{1/2}M$$

$$g_0 = Z_\Psi^{-1}Z_\phi^{-1/2}Z_g g$$

$$\lambda_0 = Z_\phi^{-2}\lambda Z_\lambda$$

we do have to add one for the κ coupling:

$$\kappa_0 = Z_\phi^{-3/2}Z_\kappa\kappa$$

What about the Z factors? Comparing the Z factors in chapter 51 to those in our solution to problem 51.3, we see that only Z_M and Z_m have changed.

Recall that the anomalous dimensions of the fields are just a derivative with respect to the Z factor for the field. Since the field Z factors haven't changed, we can quote the result from the previous problem:

$$\boxed{\gamma_\phi = \frac{g^2}{8\pi^2}}$$

$$\boxed{\gamma_\Psi = \frac{g^2}{32\pi^2}}$$

Further, we see from the above that equations 52.1 and 52.2 still hold, and also that the Z factors quoted therein haven't changed. Therefore, the beta functions are the same as in the chapter:

$$\boxed{\beta_g = \frac{5g^3}{16\pi^2}}$$

$$\boxed{\beta_\lambda = \frac{3\lambda^2 + 8\lambda g^2 - 48g^4}{16\pi^2}}$$

Now for β_κ. We take:

$$\log\left(Z_\phi^{-3/2} Z_\kappa\right) = \log\left(1 - \frac{1}{\varepsilon}\left(\frac{3mg^3}{\pi^2\kappa} - \frac{3\lambda}{16\pi^2} - \frac{3g^2}{8\pi^2}\right)\right) = \sum_{n=1}^{\infty} \frac{A_n}{\varepsilon^n} \tag{52.2.1}$$

Now we take the logarithm of our κ_0-κ relation, and shift the mass dimensionality onto $\tilde{\mu}$:

$$\log\kappa_0 = \log(Z_\phi^{-3/2} Z_\kappa) + \log\kappa + \frac{1}{2}\varepsilon\log\tilde{\mu}$$

Now we take the derivative with respect to $\ln\mu$, and recall that the bare fields must be independent of μ. We also multiply by κ. Thus:

$$0 = \sum_{n=1}^{\infty}\left(\kappa\frac{\partial A_n}{\partial g}\frac{dg}{d\log\mu} + \kappa\frac{\partial A_n}{\partial\lambda}\frac{d\lambda}{d\log\mu} + \kappa\frac{\partial A_n}{\partial\kappa}\frac{d\kappa}{d\log\mu}\right)\frac{1}{\varepsilon^n} + \frac{d\kappa}{d\log\mu} + \frac{1}{2}\varepsilon\kappa$$

In a renormalizable theory, $d\kappa/d\log\mu$ must be finite in the $\varepsilon \to 0$ limit. Thus we can write:

$$\frac{d\kappa}{d\log\mu} = -\frac{1}{2}\varepsilon\kappa + \beta_\kappa \tag{52.2.2}$$

This gives:

$$0 = \sum_{n=1}^{\infty}\left(\kappa\frac{\partial A_n}{\partial g}\frac{dg}{d\log\mu} + \kappa\frac{\partial A_n}{\partial\lambda}\frac{d\lambda}{d\log\mu} + \kappa\frac{\partial A_n}{\partial\kappa}\frac{d\kappa}{d\log\mu}\right)\frac{1}{\varepsilon^n} + \beta_\kappa$$

To determine the β function, we need terms with no εs. Thus, we keep only the first term in the sequence:

$$0 = \left(\kappa\frac{\partial A_1}{\partial g}(-\frac{1}{2}g) + \kappa\frac{\partial A_1}{\partial\lambda}(-\lambda) + \kappa\frac{\partial A_1}{\partial\kappa}(-\frac{1}{2}\kappa)\right) + \beta_\kappa$$

Thus:

$$\beta_\kappa = \frac{g\kappa}{2}\frac{\partial A_1}{\partial g} + \lambda\kappa\frac{\partial A_1}{\partial\lambda} + \frac{\kappa^2}{2}\frac{\partial A_1}{\partial\kappa}$$

Now from (52.2.1), we have:

$$A_1 = -\frac{3mg^3}{\pi^2\kappa} + \frac{3\lambda}{16\pi^2} + \frac{3g^2}{8\pi^2}$$

Combining these results, we have:

$$\boxed{\beta_\kappa = \frac{1}{16\pi^2}\left(6g^2\kappa + 3\lambda\kappa - 48mg^3\right)}$$

Now for the two anomalous dimensions. We have:

$$m_0 = Z_\Psi^{-1} Z_m m$$

$$M_0 = Z_\phi^{-1/2} Z_M^{1/2} M$$

Taking the logarithms:

$$\log m_0 = \log m + \log(Z_\Psi^{-1} Z_m)$$

$$\log M_0 = \log M + \log(Z_\phi^{-1/2} Z_M^{1/2})$$

Using the Z-factors from our solution to chapter 51.3:

$$\log m_0 = \log m + \log\left[\left(1 + \frac{g^2}{16\pi^2\varepsilon}\right)\left(1 + \frac{g^2}{8\pi^2\varepsilon}\right)\right]$$

$$\log M_0 = \log M + \log\left[\left(1 + \frac{g^2}{8\pi^2\varepsilon}\right)\left(1 + \frac{1}{32\pi^2\varepsilon}\right)\left(\frac{\kappa^2}{M^2} + \lambda - \frac{24g^2m^2}{M^2}\right)\right]$$

Now we take the derivative with respect to $\log\mu$; the left hand sides vanish since the bare fields must be independent of μ. Then:

$$\frac{d\log m}{d\log\mu} = -\frac{d}{d\log\mu}\log\left[1 + \frac{3g^2}{16\pi^2\varepsilon}\right]$$

$$\frac{d\log M}{d\log\mu} = -\frac{d}{d\log\mu}\log\left[1 + \frac{1}{32\pi^2\varepsilon}\left(\frac{\kappa^2}{M^2} + \lambda - \frac{24g^2m^2}{M^2} + 4g^2\right)\right]$$

Now we expand the logarithm, we use the chain rule, and we use some of the derivatives. Then:

$$\gamma_m = -\frac{dg}{d\log\mu}\frac{3g}{8\pi^2\varepsilon}$$

$$\gamma_M = -\left(\frac{dg}{d\log\mu}\frac{d}{dg} + \frac{d\lambda}{d\log\mu}\frac{d}{d\lambda} + \frac{d\kappa}{d\log\mu}\frac{d}{d\kappa}\right)\left[\frac{1}{32\pi^2\varepsilon}\left(\frac{\kappa^2}{M^2} + \lambda - \frac{24g^2m^2}{M^2} + 4g^2\right)\right]$$

Now we use 52.11, 52.12, and (52.2.2):

$$\boxed{\gamma_m = \frac{3g^2}{16\pi^2}}$$

$$\gamma_M = -\frac{1}{32\pi^2\varepsilon}\left[\left(-\frac{1}{2}\varepsilon g\right)\left(\frac{48gm^2}{M^2} + 8g\right) + (-\varepsilon\lambda)(1) + (-\frac{1}{2}\varepsilon\kappa)\left(\frac{2\kappa}{M^2}\right)\right]$$

which is:

$$\boxed{\gamma_M = \frac{1}{32\pi^2}\left[\frac{24g^2m^2}{M^2} + 4g^2 - \lambda + \frac{\kappa^2}{M^2}\right]}$$

Note: We found in problem 51.3 that Srednicki calculated Z_M incorrectly. Unfortunately, that error is propagated here, causing γ_M to be incorrect as well. As before, his is clearly the incorrect one, as κ has a mass dimensionality of one.

Srednicki 52.3. Consider the beta functions of equations 52.15 and 52.16.

(a) Let $\rho = \lambda/g^2$ and compute $d\rho/\log$. Express your answer in terms of g and ρ. Explain why it is better to work with g and ρ rather than g and λ.

Using the chain rule:

$$\frac{d\rho}{d\log\mu} = \frac{\partial\rho}{\partial g}\frac{dg}{d\log\mu} + \frac{\partial\rho}{\partial\lambda}\frac{d\lambda}{d\log\mu}$$

$$\frac{d\rho}{d\log\mu} = -\frac{2\lambda}{g^3}\left(-\frac{1}{2}\varepsilon g + \frac{5g^3}{16\pi^2}\right) + \frac{1}{g^2}\left(-\varepsilon\lambda + \frac{3\lambda^2 + 8\lambda g^2 - 48g^4}{16\pi^2}\right)$$

This gives:

$$\frac{d\rho}{d\log\mu} = \frac{1}{16\pi^2}\left[\frac{3\lambda^2}{g^2} - 48g^2 - 2\lambda\right]$$

Putting this in terms of g and ρ:

$$\boxed{\frac{d\rho}{d\log\mu} = \frac{3g^2}{16\pi^2}\left(\rho^2 - \frac{2}{3}\rho - 16\right)}$$

This is better because it is $\boxed{\text{separable}}$: there is no way to separate g and λ, but it is possible to separate g and ρ – and we will in fact do so below.

(b) Show that there are two fixed points, ρ_+^* and ρ_-^*, where $d\rho/d\log = 0$, and find their values.

Setting our result from part (a) equal to zero, we find:

$$\rho^2 - \frac{2}{3}\rho - 16 = 0$$

This obviously has two solutions; finding them with the quadratic equation, we have:

$$\boxed{\rho_\pm^* = \frac{1}{3} \pm \frac{\sqrt{145}}{3} = 4.34 \text{ and } -3.68}$$

Going forward, I will use the numerical values in my solution, but of course we really mean the "exact" answer with the radical.

(c) Suppose that, for some particular value of the renormalization scale μ, we have $\rho = 0$ and $g \lll 1$. What happens to ρ at much higher values of μ (but still low enough to keep $g \ll 1$)? At much lower values of μ?

This is just high school calculus. Our graph *of* $d\rho/d\log\mu$ has a zero at $\rho = 4.34$ and at $\rho = -3.43$. By choosing three test points, we find that $d\rho/d\log\mu < 0$ between those two points, and $d\rho/d\log\mu > 0$ when $\rho > 4.34$ or $\rho < -3.43$.

At $\rho = 0$, then, $d\rho/d\log\mu < 0$, ie a small change in $\log\mu$ will have the *opposite* small change in ρ. So, increasing μ by a significant amount will increase $\log\mu$ by a small amount, which will cause ρ to $\boxed{\text{decrease}}$ by a small amount. By the same logic, if μ decreases by a significant amount, ρ will $\boxed{\text{increase}}$.

Note that g is so small in both cases that we can ignore it.

Note: It is growing tedious to say so, but Srednicki's solution is again incorrect: he uses the same logic, but inexplicably comes to the opposite conclusion. Fortunately Prof. Phillip Argyres (University of Cincinatti) has an independent solution that concurs with mine, breaking the tie.

(d) Same question, but with an initial value of $\rho = 5$.

Opposite answer: as discussed in part (c), $d\rho/d\log\mu$ is positive in this region, and so ρ $\boxed{\text{increases}}$ as μ increases, and ρ $\boxed{\text{decreases}}$ as μ decreases.

(e) Same question, but with an initial value of $\rho = -5$.

Same answer as part (d): $d\rho/d\log\mu$ is positive in this region, and so ρ $\boxed{\text{increases}}$ as μ increases, and ρ $\boxed{\text{decreases}}$ as μ decreases.

(f) Find the trajectory in the (ρ, g) plane that is followed for each of the three starting points as μ is varied up and down. Put arrows on the trajectories that point in the direction of increasing μ.

Using the chain rule, we find:

$$\frac{d\rho}{d\log\mu} = \frac{d\rho}{dg}\frac{dg}{d\log\mu}$$

which is:

$$\frac{d\rho}{dg} = \frac{d\rho}{d\log\mu}\left(\frac{dg}{d\log\mu}\right)^{-1}$$

Thus:

$$\frac{d\rho}{dg} = \frac{3g^2}{16\pi^2}(\rho - 4.34)(\rho + 3.68)\frac{16\pi^2}{5g^3}$$

Cleaning up:

$$\frac{d\rho}{dg} = \frac{3}{5g}(\rho - 4.34)(\rho + 3.68)$$

Separating and integrating:

$$\int_{\rho=\rho_0}^{\rho=\rho} \frac{d\rho}{(\rho - 4.34)(\rho + 3.68)} = \frac{3}{5}\int_{g=g_0}^{g=g} \frac{1}{g}dg$$

Doing the integral, we find:

$$.207813\log\left[\frac{(4.34-\rho)(3.68+\rho_0)}{(3.68+\rho)(4.34-\rho_0)}\right] = \log\left[\frac{g}{g_0}\right]$$

We now bring the coefficient as an exponent, and equate the arguments of the logs. This gives:

$$\frac{g}{g_0} = \left[\frac{(4.34-\rho)(3.68+\rho_0)}{(4.34-\rho_0)(3.68+\rho)}\right]^{.207813}$$

We plot these for the three desired points on Mathematica, attached on last page. The answers for (c), (d), and (e) tell us how to draw the arrow: for $\rho_0 = 0$, the arrow goes against increasing ρ, ie to the left; for the other two it goes with increasing ρ, ie to the right.

(g) Explain why $\boldsymbol{\rho^*_-}$ is called an ultraviolet stable fixed point, and why $\boldsymbol{\rho^*_+}$ is called an *infrared stable* fixed point.

The answer comes from these graphs (recall that the graphs are *parameterized* by μ). If we start to the left of -3.68, the value of ρ will approach -3.68, but never go beyond it, as μ increases. This explains stable. Since the energy μ is increasing (toward, and beyond, the ultraviolet), we call this the ultraviolet stable point.

Similarly, if we start to the right of 4.34 and decrease μ, ρ will approach 4.34, which explains stable. Since the energy μ is decreasing (toward, and beyond, the infrared), we call this the infrared stable point.

Note: of course, we are using these terms in the QFT way: ultraviolet means (relatively) high energy; infrared means relatively low energy. We are making no claim that energy scales exactly correspond to those of UV or IR radiation.

Graphs for Srednicki 52.3 (f)