# Srednicki Chapter 51

QFT Problems & Solutions

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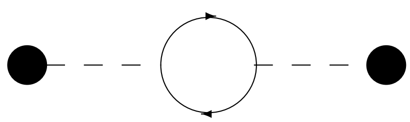
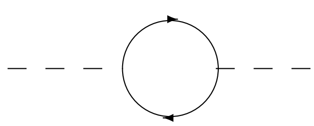
**Srednicki 51.1. Derive the fermion-loop correction to the scalar proagator by working through equation 45.2, and show that it has an extra minus sign relative to the case of a scalar loop.**

*Note: I don't like this problem very much because it is poorly explained. Given what we have done up to this point, it would be reasonable to expect the diagram in question to be the following:*



*In fact, however, we are not going to consider the lines leading to the sources as propagators; we are rather going to consider the lines* themselves *as external sources. This makes sense when phrased in this way, but this seems to contradict the instructions to follow equation 45.2. In fact we must change equation 45.2 to account for our new vertex.*

Our diagram is:



This corresponds to a vertex linking two propagators with one external $\phi$-field. We must then modify equation 45.2 to account for this:

$$Z(\eta,\overline{\eta},J) = \exp\left[ig\int d^4x\ \phi(x)\left(i\frac{\delta}{\delta\eta(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta\overline{\eta}(x)}\right)\right]\exp\left[i\int d^4xd^4y\overline{\eta}(x)S(x-y)\eta(y)\right]$$

$$\times\exp\left[\frac{i}{2}\int d^4xd^4yJ(x)\Delta(x-y)J(y)\right]$$

Now we need to write these exponents as a Taylor Series, and keep only those terms corresponding to our diagram above. In particular, we can keep only the lowest-order term (1) from the final exponential, since we have no scalar propagators (the external scalar fields do not count as propagators, as discussed above). We have two vertices and two Dirac propagators, so we'll keep only the second-order terms from the expansion of the first two exponentials. This gives:

$$Z(\eta,\overline{\eta},J) = \frac{(ig)^2}{4}\int d^4x\ \phi(x)\left(i\frac{\delta}{\delta\eta(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta\overline{\eta}(x)}\right) d^4y\ \phi(y)\left(i\frac{\delta}{\delta\eta(y)}\right)\left(\frac{1}{i}\frac{\delta}{\delta\overline{\eta}(y)}\right)$$
$$\times(i)\ d^4a\ d^4b\ \overline{\eta}(a)S(a-b)\eta(b)\ (i)\ d^4c\ d^4d\ \overline{\eta}(c)S(c-d)\eta(d)$$

Now we have to be a little bit careful. Before we could just start madly differentiating stuff, but this time we have to worry about commutation. Let's organize, and pull the scalars to the front.

$$Z(\eta,\overline{\eta},J) = \frac{g^2}{4}\int d^4x\ d^4y\ d^4a\ d^4b\ d^4c\ d^4d\ \phi(x)\phi(y)\ \left(\frac{\delta}{\delta\eta(x)}\right)\left(\frac{\delta}{\delta\overline{\eta}(x)}\right)\left(\frac{\delta}{\delta\eta(y)}\right)\left(\frac{\delta}{\delta\overline{\eta}(y)}\right)$$
$$\times\ \overline{\eta}(a)S(a-b)\eta(b)\overline{\eta}(c)S(c-d)\eta(d)$$

To start the differentiation process, let's move the last two functional derivatives past the $\overline{\eta}(a)S(a-b)$. We'll get a negative sign from the anticommutation with the functional derivative. $S(a-b)$ is made up of scalars and therefore commutes, see equation 45.4. Then:

$$Z(\eta,\overline{\eta},J) = -\frac{g^2}{4}\int d^4x\ d^4y\ d^4a\ d^4b\ d^4c\ d^4d\ \phi(x)\phi(y)\ \left(\frac{\delta}{\delta\eta(x)}\right)\left(\frac{\delta}{\delta\overline{\eta}(x)}\right)$$
$$\times\ \overline{\eta}(a)S(a-b)\left(\frac{\delta}{\delta\eta(y)}\right)\left(\frac{\delta}{\delta\overline{\eta}(y)}\right)\eta(b)\overline{\eta}(c)S(c-d)\eta(d)$$

Now we can take these two functional derivatives: recall that functional derivatives give delta functions, which we integrate over. Further, we could have chosen to use these functional derivatives on the other propagators; the result is the same up to dummy indices, so we multiply by four (two for $\eta$ and two for $\overline{\eta}$):

$$Z(\eta,\overline{\eta},J) = -g^2\int d^4x\ d^4y\ d^4a\ d^4d\ \phi(x)\phi(y)\ \left(\frac{\delta}{\delta\eta(x)}\right)\left(\frac{\delta}{\delta\overline{\eta}(x)}\right)$$
$$\times\ \overline{\eta}(a)S(a-y)S(y-d)\eta(d)$$

Now we can take the remaining functional derivatives:

$$Z(\eta,\overline{\eta},J) = -g^2\int d^4x\ d^4y\ \phi(x)\phi(y)S(x-y)S(y-x)$$

Sure enough, this has a negative sign. In the case of a scalar loop, there would be no anticommutation, and so there would be no negative sign.

Now notice that since we started with equation 45.2, we've kept only one term, but we've never formally associated that term with the diagram above. Now we use equation 45.6 to do so. The extra minus sign means that we have to build our minus sign into the value of the diagram, ie into the Feynman Rules. But from this point, the only way forward (that we've covered) is to use our Feynman rules to assess the values of the diagrams, and then to use the Lehmann-Källen form to turn the 1PI diagrams into a propagator. This is what Srednicki did in the chapter, so we are done. The result is equations 51.25 and 51.11.

**Srednicki 51.2. Finish the computation of $V_Y(p', p)$, imposing the condition $V_Y(0,0) = ig\gamma_5$.**

Start with 51.47:

$$iV_Y(p', p) = \frac{g^3}{8\pi^2}\left[\left(\frac{1}{\varepsilon} - \frac{1}{4} - \frac{1}{2}\int dF_3 \log\left(\frac{D}{\mu^2}\right)\right)\gamma_5 + \frac{1}{4}\int dF_3 \frac{\tilde{N}}{D}\right] - Z_g g\gamma_5$$

Now we need to determine $D$ and $\tilde{N}$ given $p = p' = 0$:

$$D = (x_1 + x_2)m^2 + x_3 M^2 = (1 - x_3)m^2 + x_3 M^2$$

$$\tilde{N} = m^2\gamma_5$$

$$dF_3 = 2\int_0^1 dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - 1)$$

Since there is no dependence on $x_1$ or $x_2$, we can simplify:

$$\int dF_3(\ldots) = 2\int_0^1 dx_3 \int_0^{1-x_3} dx_2(\ldots)$$

Doing the inner integral gives:

$$\int dF_3(\ldots) = 2\int_0^1 dx_3(1 - x_3)(\ldots)$$

Putting all this together gives:

$$iV_Y(0,0) = \frac{g^3}{8\pi^2}\left[\left\{\frac{1}{\varepsilon} - \frac{1}{4} - \int_0^1 dx_3(1 - x_3)\log\left(\frac{(1 - x_3)m^2 + x_3 M^2}{\mu^2}\right)\right\}\gamma_5 + \right.$$
$$\left.\frac{1}{2}\int dx_3(1 - x_3)\frac{m^2\gamma_5}{(1 - x_3)m^2 + x_3 M^2}\right] - Z_g g\gamma_5$$

Reordering:

$$iV_Y(0,0) = \frac{g^3}{8\pi^2}\left[\left\{\frac{1}{\varepsilon} - \frac{1}{4} - \int_0^1 dx_3(1 - x_3)\log\left(\frac{m^2 + x_3(M^2 - m^2)}{\mu^2}\right)\right\}\gamma_5 + \right.$$
$$\left.\frac{1}{2}\int dx_3(1 - x_3)\frac{m^2\gamma_5}{m^2 + x_3(M^2 - m^2)}\right] - Z_g g\gamma_5$$

Solving these integrals, we have:

$$iV_Y(0,0) = \frac{g^3}{8\pi^2}\left[\left\{\frac{1}{\varepsilon} - \frac{1}{4} - \log\left(\frac{M}{\mu}\right) + \frac{3}{4} + \frac{m^2}{2(M^2 - m^2)} - m^2\frac{2M^2 - m^2}{(M^2 - m^2)^2}\log\left(\frac{M}{m}\right)\right\}\gamma_5 + \right.$$

$$\frac{m^2\gamma_5 M^2\log(M/m)}{(M^2-m^2)^2}-\frac{m^2\gamma_5}{2(M^2-m^2)}\Bigg]-Z_g g\gamma_5$$

Two of these terms cancel, another two combine:

$$iV_Y(0,0)=\frac{g^3}{8\pi^2}\left[\left\{\frac{1}{\varepsilon}+\frac{1}{2}-\log\left(\frac{M}{\mu}\right)-m^2\frac{2M^2-m^2}{(M^2-m^2)^2}\log\left(\frac{M}{m}\right)\right\}\gamma_5+\right.$$

$$\left.\frac{m^2\gamma_5 M^2\log(M/m)}{(M^2-m^2)^2}\right]-Z_g g\gamma_5$$

Cancelling another two terms:

$$iV_Y(0,0)=\frac{g^3}{8\pi^2}\gamma_5\left[\frac{1}{\varepsilon}+\frac{1}{2}-\log\left(\frac{M}{\mu}\right)-m^2\frac{M^2-m^2}{(M^2-m^2)^2}\log\left(\frac{M}{m}\right)\right]-Z_g g\gamma_5$$

which is:

$$V_Y(0,0)=iZ_g g\gamma_5-i\frac{g^3}{8\pi^2}\gamma_5\left[\frac{1}{\varepsilon}+\frac{1}{2}-\log\left(\frac{M}{\mu}\right)-\frac{m^2}{M^2-m^2}\log\left(\frac{M}{m}\right)\right]$$

Now we use 51.54:

$$ig\gamma_5=iZ_g g\gamma_5-i\frac{g^3}{8\pi^2}\gamma_5\left[\frac{1}{\varepsilon}+\frac{1}{2}-\log\left(\frac{M}{\mu}\right)-\frac{m^2}{M^2-m^2}\log\left(\frac{M}{m}\right)\right]$$

which gives:

$$Z_g=1+\frac{g^2}{8\pi^2}\left[\frac{1}{\varepsilon}+\frac{1}{2}-\log\left(\frac{M}{\mu}\right)-\frac{m^2}{M^2-m^2}\log\left(\frac{M}{m}\right)\right]$$

This concurs with 51.48, a good sign. Next we have:

$$iV_Y(p',p)=\frac{g^3}{8\pi^2}\left[\left(\frac{1}{\varepsilon}-\frac{1}{4}-\frac{1}{2}\int dF_3\log\left(\frac{D}{\mu^2}\right)\right)\gamma_5+\frac{1}{4}\int dF_3\frac{\tilde{N}}{D}\right]$$

$$-\left(1+\frac{g^2}{8\pi^2}\left[\frac{1}{\varepsilon}+\frac{1}{2}-\log\left(\frac{M}{\mu}\right)-\frac{m^2}{M^2-m^2}\log\left(\frac{M}{m}\right)\right]\right)g\gamma_5$$

Simplifying:

$$iV_Y(p',p)=-g\gamma_5+\frac{g^3}{8\pi^2}\left[\left(-\frac{3}{4}-\frac{1}{2}\int dF_3\log\left(\frac{D}{\mu^2}\right)\right)\gamma_5+\frac{1}{4}\int dF_3\frac{\tilde{N}}{D}\right]$$

$$+\frac{g^3}{8\pi^2}\gamma_5\left[\log\left(\frac{M}{\mu}\right)+\frac{m^2}{M^2-m^2}\log\left(\frac{M}{m}\right)\right]$$

Now notice that in the first integral, we can write this as: $-\frac{1}{2}\int dF_3\log D+\frac{1}{2}\int dF_3\log\mu^2$. This second term is then $\log\mu^2\int_0^1(1-x)dx=\log\mu$. This then cancels with the denominator of the first term on the second row. Thus:

$$iV_Y(p',p)=-g\gamma_5+\frac{g^3}{8\pi^2}\left[\left(-\frac{3}{4}-\frac{1}{2}\int dF_3\log D+\log(M)+\frac{m^2}{M^2-m^2}\log\left(\frac{M}{m}\right)\right)\gamma_5+\frac{1}{4}\int dF_3\frac{\tilde{N}}{D}\right]$$

which is:

$$V_Y(p',p) = ig\gamma_5 + \frac{ig^3}{8\pi^2}\left[\left(\frac{3}{4}+\frac{1}{2}\int dF_3 \log D - \log(M) - \frac{m^2}{M^2-m^2}\log\left(\frac{M}{m}\right)\right)\gamma_5 - \frac{1}{4}\int dF_3 \frac{\tilde{N}}{D}\right]$$

Note that $\int dF_3$ const $=$ const. Thus, we can combine two of these terms:

$$V_Y(p',p) = ig\gamma_5 + \frac{ig^3}{8\pi^2}\left[\left(\frac{3}{4}+\frac{1}{2}\int dF_3 \log\left(\frac{D}{M^2}\right) - \frac{m^2}{M^2-m^2}\log\left(\frac{M}{m}\right)\right)\gamma_5 - \frac{1}{4}\int dF_3 \frac{\tilde{N}}{D}\right]$$

which is the final answer.

*Note: In Srednicki's solutions, the first + sign on the right hand side is a negative sign. Someone is off by a sign, I suspect it is him. He also has the $\gamma_5$ matrix distributed to the $\tilde{N}$ term, which is definitely incorrect.*

**Srednicki 51.3. Consider making $\phi$ a scalar rather than a pseudoscalar, so that the Yukawa interaction is $\mathcal{L}_{Yuk} = g\phi\overline{\Psi}\Psi$. In this case, renormalizability requires us to add a term $\mathcal{L}_{\phi^3} = \frac{1}{6}Z_\kappa\kappa\phi^3$, as well as a term linear in $\phi$ to cancel tadpoles. Find the one-loop contributions to the renormalizing Z factors for this theory in the $\overline{\text{MS}}$ scheme.**

Let's rewrite equation 51.5 (the other equations still hold, though we must add, as mentioned, a $Y\phi$ term to the counterterm Lagrangian).

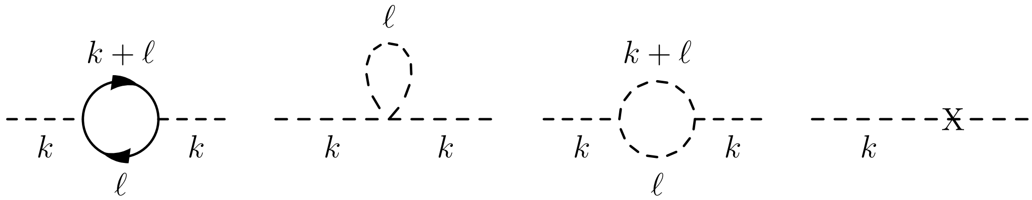
$$\mathcal{L}_1 = Z_g g\phi\overline{\Psi}\Psi + \frac{1}{6}Z_\kappa\kappa\phi^3 + \frac{1}{24}Z_\lambda\lambda\phi^4$$

These three terms represent the only possible interaction terms with mass dimensions $\leq 4$. Note that we added a renormlaization factor to the Yukawa interaction, as in the text.

Now we have seven renormalization factors (three above, and four in equation 51.6. Y is a real number, as discussed on page 66; we could solve for it, but the result would not be very interesting, and has nothing to do with the renormalizing Z factors we are asked to find). The rest of the problem is to figure out these seven Z-factors.

Recall that we are using the $\overline{\text{MS}}$ renormalization scheme, so we need only to cancel all the divergent terms. The finite terms do not affect the Z factors.

We start by trying to correct our scalar propagator at the one-loop level. The diagrams are:




Now we need to assess the value of each diagram.

Diagram 1

We have:

- (-1) because there is a Fermion loop
- $\left(\frac{1}{i}\right)^2 \tilde{S}(\not{k}+\not{\ell})\tilde{S}(\not{\ell})$ from the loop, along with an integral over $\ell$.
- $(ig)^2$ from the two vertices, since $Z_g = 1 + O(g^2)$.

Then:

$$\Pi = -g^2 \int \frac{d^4\ell}{(2\pi)^4} \tilde{S}(\not{k}+\not{\ell})\tilde{S}(\not{\ell})$$

We can take a trace over these propagators, just be writing in index notation and reordering. Thus:

$$\Pi = -g^2 \int \frac{d^4\ell}{(2\pi)^4} \operatorname{Tr}\left[\tilde{S}(\not{k}+\not{\ell})\tilde{S}(\not{\ell})\right]$$

Now we use equation 51.13. Let's look at the numerator:

$$\text{numer} = \operatorname{Tr}\left[(-\not{\ell}-\not{k}+m)(-\not{\ell}+m)\right]$$

Dropping those terms with an odd number of gamma matrices we have:

$$\text{numer} = \operatorname{Tr}\left[\not{\ell}\not{\ell}+\not{k}\not{\ell}+m^2\right]$$

which is:

$$\text{numer} = 4\left[-(\ell\cdot\ell)-(k\cdot\ell)+m^2\right]$$

Simplifying:

$$\text{numer} = 4\left[m^2-(\ell+k)\ell\right]$$

which we define to be

$$\text{numer} = 4N'$$

which we define in analogy to equation 51.14.

As for the denominator, we use equation 51.15. Putting all this together, we have:

$$i\Pi = -\frac{g^2}{4\pi^4}\int d^4\ell \int_0^1 dx \frac{N'}{(q^2+D)^2}$$

where $q = \ell + xk$ and $D = x(1-x)k^2 + m^2$.

Now let's change the integration variable $\ell \to q$.

$$i\Pi = -\frac{g^2}{4\pi^4}\int_0^1 dx\ d^4q \frac{N'}{(q^2+D)^2} \tag{51.3.1}$$

Now let's put in our N', in terms of q rather than $\ell$:

$$i\Pi = -\frac{g^2}{4\pi^4}\int_0^1 dx\ d^4q \frac{m^2 - q^2 + 2xkq - x^2k^2 - kq + xk^2}{(q^2+D)^2}$$

The terms linear in $q$ integrate to zero, so:

$$i\Pi = -\frac{g^2}{4\pi^4}\int_0^1 dx\ d^4q \frac{m^2 - q^2 - x^2k^2 + xk^2}{(q^2+D)^2}$$

Next let's make $g \to g\tilde{\mu}^{\varepsilon/2}$, shifting the mass dimensionality off of $g$.

$$i\Pi = -\frac{g^2}{4\pi^4}\tilde{\mu}^{\varepsilon}\int_0^1 dx\ d^4q \frac{m^2 - q^2 - x^2k^2 + xk^2}{(q^2+D)^2}$$

Now we use 51.18 and 51.19, and simplify:

$$\Pi = -\frac{g^2}{4\pi^2}\int_0^1 dx \left\{(m^2 - x^2k^2 + xk^2)\left(\frac{2}{\varepsilon} - \log\left(\frac{D}{\mu^2}\right)\right) + 2D\left(\frac{2}{\varepsilon} + \frac{1}{2} - \log\left(\frac{D}{\mu^2}\right)\right)\right\}$$

Now all we really care about in the $\overline{\text{MS}}$ scheme is the divergent part, so we can write this as:

$$\Pi = -\frac{g^2}{4\pi^2}\int_0^1 dx \left\{\frac{2(m^2 - x^2k^2 + kx^2) + 4D}{\varepsilon} + (\text{finite})\right\}$$

Now let's put the D in, and simplify:

$$\Pi = -\frac{g^2}{4\pi^2}\int_0^1 dx \left\{\frac{6m^2 + 6xk^2 - 6k^2x^2}{\varepsilon} + (\text{finite})\right\}$$

Doing the integral:

$$\Pi = -\frac{g^2}{4\pi^2}\left\{\frac{6m^2 + k^2}{\varepsilon} + (\text{finite})\right\}$$

Diagram 2

Nothing has changed from the text, so we can quote the answer (equation 51.21):

$$\Pi = \frac{\lambda}{(4\pi)^2}\left[\frac{1}{\varepsilon} + \frac{1}{2} - \frac{1}{2}\log\left(\frac{M^2}{\mu^2}\right)\right]M^2$$

Dropping the finite part:

$$\Pi = \frac{\lambda}{(4\pi)^2}\left[\frac{1}{\varepsilon} + (\text{finite})\right]M^2$$

Diagram 3

This is the same diagram as in $\phi^3$ theory. However, we cannot quote the answer from section 14, because that was done in six dimensions, but now we are working in four dimensions. We can start from scratch, but everything we did up to equation (51.3.1) still holds with the following modifications:

- There is no negative sign, since there is no fermion loop
- The vertex factor is $i\kappa$
- There is a symmetry factor of 2
- $g \to g\tilde{\mu}^{\varepsilon/2}$

This gives:

$$i\Pi = \frac{\kappa^2\tilde{\mu}^\varepsilon}{2}\int_0^1 dx \frac{d^4q}{(2\pi)^4}\frac{1}{(q^2+D)^2}$$

Using equation 51.18:

$$\Pi = \frac{\kappa^2}{2}\int_0^1 dx \frac{1}{16\pi^2}\left(\frac{2}{\varepsilon} - \log\left(\frac{D}{\mu^2}\right)\right)$$

This is:

$$\Pi = \frac{\kappa^2}{16\pi^2\varepsilon} + (\text{finite})$$

Diagram 4

This is just a vertex, so we can take the value straight from the chapter 45 Feynman Rules (or more recently, equation 51.22).

$$\Pi = -(Z_\phi - 1)k^2 - (Z_M - 1)M^2$$

□

Now we sum all these self-energies, and choose the $Z$s to cancel the infinite parts.

$$-(Z_\phi - 1)k^2 - \frac{g^2k^2}{4\pi^2\varepsilon} = 0$$

which implies:

$$\boxed{Z_\phi = 1 - \frac{g^2}{4\pi^2\varepsilon}}$$

Similarly:

$$-(Z_M - 1)M^2 + \frac{\kappa^2}{16\pi^2\varepsilon} + \frac{\lambda}{16\pi^2\varepsilon}M^2 - \frac{g^2 6m^2}{4\pi^2\varepsilon} = 0$$

Dividing through by $M^2$:

$$-(Z_M - 1) + \frac{\kappa^2}{16\pi^2\varepsilon M^2} + \frac{\lambda}{16\pi^2\varepsilon} - \frac{g^2 6m^2}{4\pi^2 M^2\varepsilon} = 0$$

Solving this:

$$\boxed{Z_M = 1 + \frac{1}{16\pi^2\varepsilon}\left[\frac{\kappa^2}{M^2} + \lambda - \frac{24g^2m^2}{M^2}\right]}$$

*Note: this is different from Srednicki's solution, but it agrees with an independent solution from Andre Schneider at the University of Indiana. Moreover, Srenicki's solution is necessarily incorrect because the mass $\phi^3$ vertex forces $\kappa$ to have a mass dimension of 1; Srednicki's solution therefore adds a term with mass dimension = 2 to terms of mass dimension = 0, which is obviously wrong.*

Now for the fermion propagator. The diagrams are:

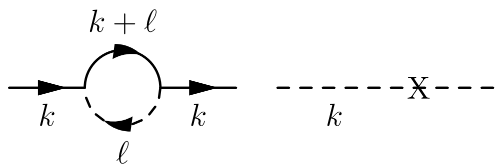



Diagram 2

Using the counterterm Lagrangian, we change the $\partial\!\!\!/$ to $ik\!\!\!/$ and rub out the fields, we get the vertex factor for the counterterm vertex (don't forget to multiply by $i$, as with all vertex factors) as:

$$i\Pi = -i(Z_\Psi - 1)k\!\!\!/ - i(Z_m - 1)m$$

Then:

$$\Pi = -(Z_\Psi - 1)k\!\!\!/ + (Z_m - 1)m$$

Diagram 1

Diagram one has two vertices, two propagators, and an integral over the loop; the result is:

$$i\Pi = (ig)^2\left(\frac{1}{i}\right)^2\int\frac{d^4\ell}{(2\pi)^4}\tilde{S}(k\!\!\!/ + \ell\!\!\!/)\tilde{\Delta}(\ell^2)$$

which is:

$$i\Pi = g^2\int\frac{d^4\ell}{(2\pi)^4}\frac{-k\!\!\!/ - \ell\!\!\!/ + m}{[(k+\ell)^2+m^2][\ell^2+m^2]}$$

Using 51.15 to combine these denominators:

$$i\Pi = g^2\int\frac{d^4\ell}{(2\pi)^4}\int_0^1 dx\frac{-k\!\!\!/ - \ell\!\!\!/ + m}{(q^2+D)^2}$$

where $q = \ell + xk$ and $D = x(1-x)k^2 + m^2$.

Now we change the integration variable $\ell \to q$. Thus:

$$i\Pi = g^2\int\frac{d^4q}{(2\pi)^4}\int_0^1 dx\frac{-k\!\!\!/ - q\!\!\!/ + xk\!\!\!/ + m}{(q^2+D)^2}$$

We can drop the terms in the numerator that are odd in $q$, since those integrate to zero. Further, we take $g \to g\mu^{\varepsilon/2}$, shifting the mass dimensionality onto $\tilde{\mu}$. Then:

$$i\Pi = g^2\mu^\varepsilon \int \frac{d^4q}{(2\pi)^4} \int_0^1 dx \frac{m + \not{k}(x-1)}{(q^2+D)^2}$$

We use equation 51.18 to do the q-integral:

$$i\Pi = \frac{ig^2}{16\pi^2} \int_0^1 dx \left[m + \not{k}(x-1)\right] \left[\frac{2}{\varepsilon} - \log\left(\frac{D}{\mu^2}\right)\right]$$

Doing the x integral:

$$i\Pi = \frac{ig^2}{16\pi^2} \left[m - \frac{\not{k}}{2}\right] \left[\frac{2}{\varepsilon} - \log\left(\frac{D}{\mu^2}\right)\right]$$

This is:

$$\Pi = \frac{g^2}{8\pi^2\varepsilon} \left(m - \frac{\not{k}}{2}\right) + (\text{finite})$$

□

Combining these two diagrams, and requiring the $m$ terms to cancel, we have:

$$\frac{1}{\varepsilon}\frac{g^2}{8\pi^2} m - (Z_m - 1)m = 0$$

Thus:

$$\boxed{Z_m = 1 + \frac{g^2}{8\pi^2\varepsilon}}$$

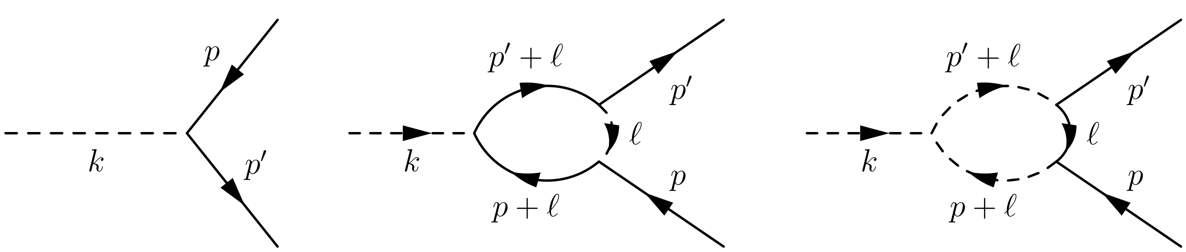
and similarly with the $\not{k}$ terms:

$$-(Z_\Psi - 1)\not{k} - \frac{\not{k}}{2}\frac{1}{\varepsilon}\frac{g^2}{8\pi^2} = 0$$

Thus:

$$\boxed{Z_\Psi = 1 - \frac{g^2}{16\pi^2\varepsilon}}$$

Now we consider the correction to the $\phi\overline{\Psi}\Psi$ vertex. The diagrams are:




Assessing the values of these diagrams, we have:

Diagram 1

The only contribution comes from the vertex, so:

$$i\Pi = iZ_g g$$

so:

$$\Pi = Z_g g$$

Diagram 3

Note that there is only one fermion propagator, so the numerator will at most have terms of only $O(\ell)$. There are three total propagators, so the denominator will be of $O(\ell^6)$. Thus, this integral goes as $\int d\ell^4 \frac{1}{\ell^5}$. Even after four integrals, this will remain convergent assuming reasonable boundary conditions. This therefore does not contribute to the Z-factors in this renormaliation scheme.

Diagram 2

Assessing the value of this diagram, we have:

$$i\Pi = (ig)^3 \left(\frac{1}{i}\right)^3 \int \frac{d^4\ell}{(2\pi)^4} \tilde{S}(\not{p}' + \not{\ell})\tilde{S}(\not{p} + \not{\ell})\Delta(\ell^2)$$

Writing these propagators:

$$i\Pi = g^3 \int \frac{d^4\ell}{(2\pi)^4} \frac{(-\not{p}' - \not{\ell} + m)(-\not{p} - \not{\ell} + m)}{((p+\ell)^2 + m^2))\,((p+\ell)^2 + m^2)\,(\ell^2 M^2)}$$

Comparing this to equation 51.41, only the definition of $N$ has changed. Thus:

$$i\Pi = g^3 \int \frac{d^4\ell}{(2\pi)^4} \int dF_3 \frac{N}{(q^2 + D)^3}$$

We could calculate N, but this is a lot of messy algebra. Recall that we only care about the divergent part. By equation 14.27, the only divergent part has a $q^2$ in the numerator. We have:

$$N \propto \not{\ell}^2 \propto \not{q}^2 = -q^2$$

Also, we shift the mass dimensionality onto g: $g \to g\mu^{\varepsilon/2}$. This gives:

$$i\Pi = g^3 \mu^{\varepsilon/2} \int \frac{d^4\ell}{(2\pi)^4} \int dF_3 \frac{q^2}{(q^2 + D)^3} + (\text{finite})$$

Now we use 14.27 to solve the integral (recall that the dimensionality is $4 - \varepsilon$).. We also perform a Wick Rotation, which adds a factor of $i$. This gives:

$$i\Pi = -\frac{ig^3}{(4\pi)^2} \int dF_3 \Gamma\left(\frac{\varepsilon}{2}\right) + (\text{finite})$$

which is:

$$\Pi = -\frac{g^3}{8\pi^2\varepsilon}\int dF_3 + (\text{finite})$$

Using equation 14.11, we have:

$$\Pi = -\frac{g^3}{8\pi^2\varepsilon} + (\text{finite})$$

□

Putting these together, we have:

$$iV_Y = iZ_g g - \frac{g^3}{8\pi^2\varepsilon} + (\text{finite})$$

Choosing $Z_g$ to absorb this divergence, we achieve, up to order $g^2$:

$$\boxed{Z_g = 1 + \frac{g^2}{8\pi^2\varepsilon}}$$

Next we turn to the $\phi^3$ vertex. The diagrams are:

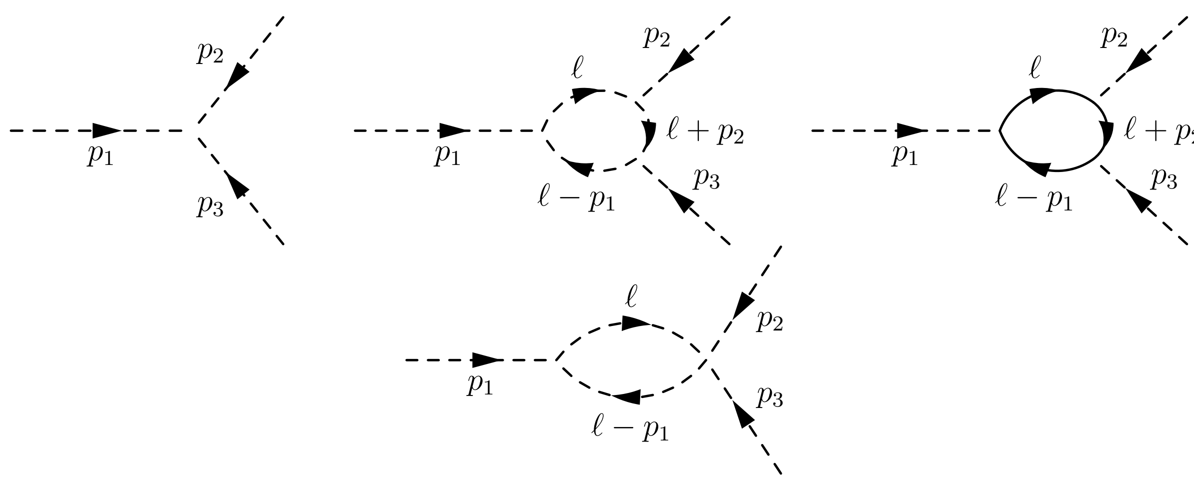



Diagram 1

The only contribution comes from the vertex factor. Dropping the $i$, we have:

$$\Pi = Z_\kappa \kappa$$

Diagram 2

This is the same diagram as in $\phi^3$ theory. Since we are now working in four dimensions, we cannot simply quote the result from section 16. We can, however, use equation 16.6:

$$\Pi = g^2 \int dF_3 \int \frac{d^d\overline{q}}{(2\pi)^d}\frac{1}{(\overline{q}^2 + D)^3}$$

Equation 16.7 gives:

$$\Pi = g^2 \int dF_3 \frac{\Gamma(3-\frac{d}{2})}{2(4\pi)^{d/2}} D^{-(3-d/2)}$$

Now take $d = 4 - \varepsilon$:

$$\Pi = g^2 \int dF_3 \frac{\Gamma\left(1+\frac{\varepsilon}{2}\right)}{2(4\pi)^2} D^{-1+\varepsilon/2}$$

Taking $\varepsilon \to 0$:

$$\Pi = g^2 \int dF_3 \frac{1}{2(4\pi)^2} D^{-1}$$

Now we use 16.5:

$$\Pi = \frac{g^2}{2(4\pi)^2} \int dF_3 \left[x_3 x_1 k_1^2 + x_3 x_2 k_2^2 + x_1 x_2 k_3^2 + m^2\right]^{-1}$$

Now we can set the external momenta equal to zero, since they don't contribute to the divergent part (actually we can set all of this equal to zero, since it is already manifestly convergent, but let's stick with our usual set of tricks):

$$\Pi = \frac{g^2}{32\pi^2 m^2}$$

Dropping the convergent part, we have:

$$\Pi = \text{(finite)}$$

<u>Diagram 3</u>

Here we have to start from scratch. We have:

$$i\Pi = (-1)(ig)^3 \left(\frac{1}{i}\right)^3 \int \frac{d^4\ell}{(2\pi)^4} \tilde{S}(\ell)\tilde{S}(p_2+\ell)\tilde{S}(\ell-p_1)$$

where the negative sign is necessary because we have a fermion loop. Expanding this:

$$i\Pi = -g^3 \int \frac{d^4\ell}{(2\pi)^4} \frac{(-\not{\ell}+m)(-\not{p}_2-\not{\ell}+m)(-\not{\ell}+\not{p}_1+m)}{[(\ell-p_1)^2+m^2][(p_2+\ell)^2+m^2][\ell^2+m^2]}$$

Consider the numerator. We set the external momenta to zero. The terms odd in $\ell$ integrate to zero, so we have:

$$\text{numer} = 3m\not{\ell}\not{\ell} + m^3$$

Using our usual trick (Feynman's formula), we can combine the denominator. Then:

$$i\Pi = -g^3 \int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dF_3 \frac{-12m\ell^2 + m^3}{(q^2+D)^3}$$

Switching our integration variable to $q$:

$$i\Pi = -g^3 \int \frac{d^4q}{(2\pi)^4} \int_0^1 dF_3 \frac{-12mq^2 + (\text{finite term of O}(q^0))}{(q^2+D)^3}$$

The constant term is of $O(q^{-6})$, which will clearly be convergent. Then:

$$i\Pi = 12mg^3 \int \frac{d^4q}{(2\pi)^4} \int_0^1 dF_3 \frac{q^2}{(q^2+D)^3}$$

Now we take a Wick Rotation and use equation 14.27:

$$i\Pi = 12mg^3 i \int_0^1 dF_3 \frac{\Gamma\left(\frac{\varepsilon}{2}\right)\Gamma(3)}{(4\pi)^2\Gamma(3)\Gamma(2)} D^{-\varepsilon/2}$$

Keeping the divergent terms only, we have:

$$i\Pi = 12mg^3 i \int_0^1 dF_3 \frac{1}{(4\pi)^2}\left[\frac{2}{\varepsilon} + (\text{finite})\right]$$

which is:

$$\Pi = \frac{24mg^3}{16\pi^2\varepsilon}\int_0^1 dF_3 + (\text{finite})$$

Doing the integral:

$$\Pi = \frac{3mg^3}{\pi^2\varepsilon} + (\text{finite})$$

Diagram 4

There are actually three such diagrams: due to the two different vertices, the choice of which vertex goes on the left is a substantial difference. Due to the loop, each diagram has a symmetry factor of two. Swapping the two external propagators on the right does not yield a substantively different diagram. Thus, we assess the values of these diagrams at:

$$i\Pi = \frac{3!}{2}(i\kappa)(-i\lambda)\left(\frac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2+M^2)((\ell+p_1)^2+M^2)}$$

Our usual manipulations give:

$$i\Pi = -3\kappa\lambda \int \frac{d^4\ell}{(2\pi)^4}\int_0^1 \frac{1}{(q^2+D)^2} dF_3$$

Using equation 51.18, we have:

$$i\Pi = -\frac{3i\kappa\lambda}{(4\pi)^2}\int dF_3 \left[\frac{2}{\varepsilon} - \log\left(\frac{D}{\mu^2}\right)\right]$$

which is:

$$\Pi = -\frac{3\kappa\lambda}{16\pi^2}\left[\frac{1}{\varepsilon} + (\text{finite})\right]$$

□

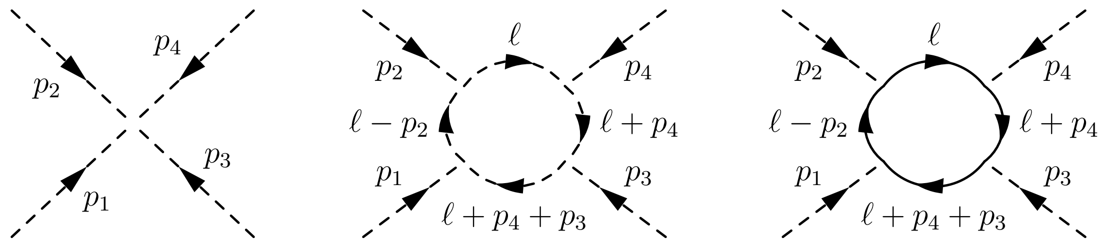
Now we combine these:

$$Z_\kappa \kappa + \frac{3mg^3}{\pi^2\varepsilon} - \frac{3\kappa\lambda}{16\pi^2\varepsilon}$$

We want $Z_\kappa$ to absorb these divergences only. These give the $O(\varepsilon^{-1})$ contributions to $Z_\kappa$:

$$\boxed{Z_\kappa = 1 + \frac{1}{\varepsilon}\left(\frac{3\lambda}{16\pi^2} - \frac{3mg^3}{\pi^2\kappa}\right)}$$

Finally, we have the $\phi^4$ vertex. The diagrams are:




In addition, there are the diagrams of figure 31.5.

Diagram 1

Only the vertex factor contributes.

$$\Pi = -Z_\lambda \lambda$$

Diagram 3

Fortunately, this is exactly the same diagram that was considered in the chapter. The factors of $\gamma_5$ in the numerator are different, but since we only care about determining the divergent part – which Srednicki tells us is $\propto \ell^4$, none of this matters. We could introduce a negative sign because we do not need to anticommute $\gamma_5$ across $\tilde{S}$, but since this happens twice, any effect will be lost. We can therefore quote Srednicki's result, equation 51.50:

$$\Pi = -\frac{3g^4}{\pi^2}\left(\frac{1}{\varepsilon} + (\text{finite})\right)$$

Diagrams of figure 31.5

Fortunately for us, $\phi^4$ theory naturally uses four dimensions, and so we can quote equation 51.51:

$$\Pi = \frac{3\lambda^2}{16\pi^2}\left(\frac{1}{\varepsilon} + (\text{finite})\right)$$

Diagram 2

Here we will have terms of $O(\ell^0)$ in the numerator since these are all scalars, but of $O(\ell^8)$ in the denominator. Even after four integrals, this will be convergent. Then:

$$\Pi = (\text{finite})$$

$\square$

We combine these in the usual way:

$$\Pi = -Z_\lambda \lambda - \frac{3g^4}{\pi^\varepsilon} + \frac{3\lambda^2}{16\pi^2 \varepsilon}$$

Choosing the contributions to $Z_\lambda$ at $O(\varepsilon^{-1})$ to cancel these divergences, we have:

$$\boxed{Z_\lambda = 1 + \frac{1}{\varepsilon}\left(\frac{3\lambda}{16\pi^2} - \frac{3g^4}{\lambda\pi^2}\right)}$$