Srednicki Chapter 50

QFT Problems & Solutions

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Srednicki 50.1. Consider a bra-ket notation for twistors,

$$|\boldsymbol{p}] = \boldsymbol{u}_-(\vec{\boldsymbol{p}}) = \boldsymbol{v}_+(\vec{\boldsymbol{p}})$$

$$|\boldsymbol{p}\rangle = \boldsymbol{u}_+(\vec{\boldsymbol{p}}) = \boldsymbol{v}_-(\vec{\boldsymbol{p}})$$

$$[\boldsymbol{p}| = \overline{\boldsymbol{u}}_+(\vec{\boldsymbol{p}}) = \overline{\boldsymbol{v}}_-(\vec{\boldsymbol{p}})$$

$$\langle\boldsymbol{p}| = \overline{\boldsymbol{u}}_-(\vec{\boldsymbol{p}}) = \overline{\boldsymbol{v}}_+(\vec{\boldsymbol{p}})$$

We then have

$$\langle\boldsymbol{k}||\boldsymbol{p}\rangle = \langle\boldsymbol{kp}\rangle$$

$$[\boldsymbol{k}||\boldsymbol{p}] = [\boldsymbol{kp}]$$

$$\langle\boldsymbol{k}||\boldsymbol{p}] = \boldsymbol{0}$$

$$[\boldsymbol{k}||\boldsymbol{p}\rangle = \boldsymbol{0}$$

(a) Show that

$$-\not{\boldsymbol{p}} = |\boldsymbol{p}\rangle[\boldsymbol{p}| + |\boldsymbol{p}]\langle\boldsymbol{p}|$$

where p is any massless four-momentum.

Let's write this as:

$$-\not{p} = \frac{1}{2}\left(1+\gamma_5+1-\gamma_5\right)(-\not{p})$$

Then:

$$-\not{p} = \frac{1}{2}(1+\gamma_5)(-\not{p}) + \frac{1}{2}(1-\gamma_5)(-\not{p})$$

Using 50.1:

$$-\not{p} = u_+(\vec{p})\overline{u}_+(\vec{p}) + u_-(\vec{p})\overline{u}_-(\vec{p})$$

which is:

$$-\not{p} = |p\rangle[p| + |p]\langle p|$$

(b) Use this notation to rederive equations 50.28-50.30

We have:

$$\overline{u}_+(\vec{k}')(-\not{k})u_+(\vec{k}) = [k'|\left[|k\rangle[k| + |k]\langle k|\right]|k\rangle$$

Distributing:

$$\overline{u}_+(\vec{p}')(-\not{k})u_+(\vec{p}) = [p'||k\rangle[k||p\rangle + [p'||k]\langle k||p\rangle$$

Using 50.34:

$$\boxed{\overline{u}_+(\vec{p}')(-\not{k})u_+(\vec{p}) = [p'k]\langle kp\rangle}$$

Similarly:

$$\overline{u}_-(\vec{k}')(-\not{k})u_-(\vec{k}) = \langle p'|\,[|k\rangle[k| + |k]\langle k|]\,|p]$$

Distributing:

$$\overline{u}_-(\vec{p}')(-\not{k})u_-(\vec{p}) = \langle p'||k\rangle[k||p] + \langle p'||k]\langle k||p]$$

Using 50.34:

$$\boxed{\overline{u}_-(\vec{p}')(-\not{k})u_-(\vec{p}) = \langle p'k\rangle[kp]}$$

And:

$$\overline{u}_-(\vec{k}')(-\not{k})u_+(\vec{k}) = \langle p'|\,[|k\rangle[k| + |k]\langle k|]\,|p\rangle$$

Distributing:

$$\overline{u}_-(\vec{p}')(-\not{k})u_+(\vec{p}) = \langle p'||k\rangle[k||p\rangle + \langle p'||k]\langle k||p\rangle$$

Using 50.34:

$$\boxed{\overline{u}_-(\vec{p}')(-\not{k})u_+(\vec{p}) = 0}$$

Finally:

$$\overline{u}_+(\vec{k}')(-\not{k})u_-(\vec{k}) = [k'|\,[|k\rangle[k| + |k]\langle k|]\,|k]$$

Distributing:

$$\overline{u}_+(\vec{p}')(-\not{k})u_-(\vec{p}) = [p'||k\rangle[k||p] + [p'||k]\langle k||p]$$

Using 50.34:

$$\boxed{\overline{u}_+(\vec{p}')(-\not{k})u_-(\vec{p}) = 0}$$

Srednicki 50.2. (a) Use equations 50.9 and 50.15 to verify equation 50.12.

Using 50.9 and 50.15, 50.12 becomes:

$$\begin{pmatrix} -p^0+p^3 & p^1-ip^2 \\ p^1+ip^2 & -p^0-p^3 \end{pmatrix} \stackrel{?}{=} -\sqrt{2\omega}\begin{pmatrix} -\sin(\theta/2)e^{-i\phi} \\ \cos(\theta/2) \end{pmatrix}\sqrt{2\omega}\begin{pmatrix} -\sin(\theta/2)e^{i\phi} & \cos(\theta/2) \end{pmatrix}$$

Multiplying:

$$\begin{pmatrix} -p^0+p^3 & p^1-ip^2 \\ p^1+ip^2 & -p^0-p^3 \end{pmatrix} \stackrel{?}{=} -2\omega\begin{pmatrix} \sin^2\frac{\theta}{2} & -\sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{-i\phi} \\ -\sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{i\phi} & \cos^2\frac{\theta}{2} \end{pmatrix}$$

Using half-angle identities:

$$\begin{pmatrix} -p^0+p^3 & p^1-ip^2 \\ p^1+ip^2 & -p^0-p^3 \end{pmatrix} \stackrel{?}{=} -\begin{pmatrix} \omega-\omega\cos\theta & -\omega\sin\theta e^{-i\phi} \\ -\omega\sin\theta e^{i\phi} & \omega+\omega\cos\theta \end{pmatrix}$$

Using Euler's formula:

$$\begin{pmatrix} -p^0+p^3 & p^1-ip^2 \\ p^1+ip^2 & -p^0-p^3 \end{pmatrix} \stackrel{\checkmark}{=} \begin{pmatrix} -\omega+\omega\cos\theta & \omega\sin\theta\cos\phi-i\omega\sin\theta\sin\phi \\ \omega\sin\theta\cos\phi+i\omega\sin\theta\sin\phi & -\omega-\omega\cos\theta \end{pmatrix}$$

(b) Let the three-momentum $\vec{p}$ be in the $+\hat{z}$ direction. Use equation 38.12 to compute $u_\pm(\vec{p})$ explicitly in the massless limit (corresponding to the limit $\eta \to \infty$, where $\sinh\eta = |\vec{p}|/m$). Verify that, when $\theta = 0$, your results agree with equations 50.8, 50.9, and 50.13.

Equation 38.12 gives:

$$u_s(\vec{p}) = \exp\left[i\eta\vec{p}\cdot\vec{K}\right] u_s(0)$$

Taking $\vec{p} = \hat{z}$,

$$u_s(\hat{z}) = \exp\left(i\eta\frac{i}{2}\gamma^3\gamma^0\right) u_s(0)$$

This simplifies to:

$$u_s(\hat{z}) = \exp\left[-\frac{\eta}{2}\mathrm{diag}(1,-1,-1,1)\right] u_s(0)$$

Now we write the series expansion of the exponential. Note that the matrix squared gives the identity. Then:

$$u_s(\hat{z}) = \left[\cosh\left(\frac{\eta}{2}\right) + \sinh\left(\frac{\eta}{2}\right)\ \mathrm{diag}(-1,1,1,-1)\right] u_s(0)$$

Clearly in the low-mass limit, this diverges. This is not very interesting, so let's follow this for a while and see if we can get something finite. First, notice that $\cosh^2 x - \sinh^2 x = 1$; in the low-mass limit, the 1 is inconsequential, and $\cosh x = \sinh x$. Then:

$$u_s(\hat{z}) = \sinh\left(\frac{\eta}{2}\right)\left[1 + \ \mathrm{diag}(-1,1,1,-1)\right] u_s(0)$$

which is:

$$u_s(\hat{z}) = \sinh\left(\frac{\eta}{2}\right)\ \mathrm{diag}(0,2,2,0) u_s(0)$$

Now we have $\sinh\frac{\eta}{2} = \sinh\left(\frac{\sinh^{-1}(p/m)}{2}\right)$. Now we use a rather obscure identity:

$$\sinh\frac{\eta}{2} = \frac{p/2}{\sqrt{2}\sqrt{\sqrt{\left(\frac{p}{m}\right)^2+1}+1}} \to \sqrt{\frac{p}{2m}}$$

where the arrow represents the massless limit. Then:

$$u_s(\hat{z}) = \sqrt{\frac{p}{2m}}\ \mathrm{diag}(0,2,2,0) u_s(0)$$

and so:

$$u_s(\hat{z}) = \sqrt{\frac{2p}{m}}\ \mathrm{diag}(0,1,1,0) u_s(0)$$

Using 38.6, and $p = E$ for a massless particle:

$$u_+(\hat{z}) = \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Similarly:

$$u_-(\hat{z}) = \sqrt{2E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

This manifestly agrees with 50.8, 50.9 and 50.13 for $\theta = 0$.

Srednicki 50.3. Prove the Schouten identity,

$$\boldsymbol{\langle pq\rangle\langle rs\rangle + \langle pr\rangle\langle sq\rangle + \langle ps\rangle\langle qr\rangle = 0}$$

See 50.13 and 50.22; each label corresponds to one twistor, and each twistor corresponds (by 50.8-50.9) to two components. Therefore, there are at most two linearly independent twistors. It is therefore not possible to have three anti-symmetric twistors, and the entire term must vanish.

All we have to do is show that the left hand side is indeed anti-symmetric. But this is trivial; swapping any two indices and using 50.21 gives the same thing with a negative sign.

Srenicki 50.4. Show that

$$\boldsymbol{\langle pq\rangle[qr]\langle rs\rangle[sp] = \mathrm{Tr}\,\frac{1}{2}(1-\gamma_5)\not{p}\not{q}\not{r}\not{s}}$$

and evaluate the right hand side.

We begin with equation 50.1, then use 50.33:

$$-\frac{1}{2}(1-\gamma_5)\not{p} = u_-(\vec{p})\overline{u}_-(\vec{p}) = |p]\langle p|$$

Now we multiply on the right by $-\not{q}$, and use 50.35:

$$\frac{1}{2}(1-\gamma_5)\not{p}\not{q} = |p]\langle p|\left(|q\rangle[q| + |q]\langle q|\right)$$

Using 50.34:

$$\frac{1}{2}(1-\gamma_5)\not{p}\not{q} = |p]\langle pq\rangle[q|$$

Now we do the same thing with $\not{r}$ and $\not{s}$. Then:

$$\frac{1}{2}(1-\gamma_5)\not{p}\not{q}\not{r}\not{s} = |p]\langle pq\rangle[qr]\langle rs\rangle[s$$

Taking the trace gives:

$$\mathrm{Tr}\,\frac{1}{2}(1-\gamma_5)\not{p}\not{q}\not{r}\not{s} = \mathrm{Tr}\;|p]\langle pq\rangle[qr]\langle rs\rangle[s$$

Now we use the cyclic property of the trace. This yields scalars on the right hand side, and so the trace drops out:

$$\text{Tr } \frac{1}{2}(1-\gamma_5)\not{p}\not{q}\not{r}\not{s} = \langle pq\rangle[qr]\langle rs\rangle[sp]$$

Rewriting slightly:

$$\boxed{\langle pq\rangle[qr]\langle rs\rangle[sp] = \text{Tr } \frac{1}{2}(1-\gamma_5)\not{p}\not{q}\not{r}\not{s}}$$

as expected. Now we rewrite the trace so that we can solve it:

$$\langle pq\rangle[qr]\langle rs\rangle[sp] = \frac{1}{2}\text{Tr } \not{p}\not{q}\not{r}\not{s} - \frac{1}{2}\text{Tr } \gamma_5\not{p}\not{q}\not{r}\not{s}$$

and use 47.13 and 47.17:

$$\boxed{\langle pq\rangle[qr]\langle rs\rangle[sp] = 2\left((ps)(qr)-(pr)(qs)+(pq)(rs)\right) + 2ip_\alpha q_\beta r_\gamma s_\delta \varepsilon^{\alpha\beta\gamma\delta}}$$

Srednicki 50.5. (a) Prove the useful identities

$$\boldsymbol{\langle p|\gamma^\mu|k] = [k|\gamma^\mu|p\rangle}$$

$$\boldsymbol{\langle p|\gamma^\mu|k]^* = \langle k|\gamma^\mu|p]}$$

$$\boldsymbol{\langle p|\gamma^\mu|p] = 2p^\mu}$$

$$\boldsymbol{\langle p|\gamma^\mu|k\rangle = 0}$$

$$\boldsymbol{[p|\gamma^\mu|k] = 0}$$

As far as I know, the only way to solve this is to multiply through by an arbitrary massless four-vector, we'll call it q_μ. Note that we have to be careful not to lose generality: for any particular q_μ, or if p^μ were orthogonal to alll q_μs, then we would have an issue. Fortunately, there exists no p_μ that is orthogonal to all q_μ, and so we have:

$$\langle p|\not{q}|k] \stackrel{?}{=} [k|\not{q}|p\rangle$$

$$\left(\langle p|\not{q}|k]\right)^* \stackrel{?}{=} \langle k|\not{q}|p]$$

$$\langle p|\not{q}|p] \stackrel{?}{=} 2(p\cdot q)$$

$$\langle p|\not{q}|k\rangle \stackrel{?}{=} 0$$

$$[p|\not{q}|k] \stackrel{?}{=} 0$$

Now we use 50.35:

$$\langle p|\left(|q\rangle[q| + |q]\langle q|\right)|k] \stackrel{?}{=} [k|\left(|q\rangle[q| + |q]\langle q|\right)|p\rangle$$

$$\left(\langle p|\left(|q\rangle[q| + |q]\langle q|\right)|k]\right)^* \stackrel{?}{=} \langle k|\left(|q\rangle[q| + |q]\langle q|\right)|p]$$

$$\langle p|\left(|q\rangle[q| + |q]\langle q|\right)|p] \stackrel{?}{=} 2(p\cdot q)$$

$$\langle p|\left(|q\rangle[q| + |q]\langle q|\right)|k\rangle \stackrel{?}{=} 0$$

$$[p|\left(|q\rangle[q| + |q]\langle q|\right)|k] \stackrel{?}{=} 0$$

Many of these terms vanish by 50.34:

$$\langle pq\rangle[qk] \stackrel{?}{=} [kq]\langle qp\rangle$$

$$(\langle pq\rangle[qk])^* \stackrel{?}{=} \langle kq\rangle[qp]$$

$$\langle pq\rangle[qp] \stackrel{?}{=} 2(p\cdot q)$$

$$0 \stackrel{\checkmark}{=} 0$$

$$0 \stackrel{\checkmark}{=} 0$$

Now we use 50.17 and 50.21:

$$\langle pq\rangle[qk] \stackrel{\checkmark}{=} [qk]\langle pq\rangle$$

$$(\langle pq\rangle[qk])^* \stackrel{?}{=} \langle qk\rangle[pq]$$

$$\langle pq\rangle[qp] \stackrel{?}{=} 2(p\cdot q)$$

$$0 \stackrel{\checkmark}{=} 0$$

$$0 \stackrel{\checkmark}{=} 0$$

As for the remaining equalities, the second is true by 50.20 and the third is true by 50.24.

(b) Extend the last two identities of part (a): show that the product of an odd number of gamma matrices sandwiched between either $\langle p|$ and $|k\rangle$ or $[p|$ and $|k]$ vanishes. Also show that the product of an even number of gamma matrices between either $\langle p|$ and $|k\rangle$ vanishes.

As before, we multiply each gamma matrix by an arbitrary massless four vector, then use equation 50.35. Each use of 50.35 will change the open grouping symbol on the left from a bracket to a $\langle$, or vice versa. If the grouping symbols start out compatible, equation 50.35 must be used an even number of times to maintain this compatibility; an odd number of applications of 50.35 will lead to $[\rangle$ or $\langle]$, which vanishes. Similarly, if the grouping symbols start out incompatible, equation 50.35 must be used an odd number of times to rectify this; using it an even number of times will again lead to a vanishing term.

(c) Prove the Fierz Identities

$$-\frac{1}{2}\langle p|\gamma_\mu|q]\gamma^\mu = |q]\langle p| + |p\rangle[q|$$

$$-\frac{1}{2}[p|\gamma_\mu|q\rangle\gamma^\mu = |q\rangle[p| + |p]\langle q|$$

Now take the matrix element of equation 50.44 between $\langle r|$ and $|s]$ to get another useful form of the Fierz identity:

Let's do both terms simultaneously. We start by using 50.33:

$$-\frac{1}{2}\langle p|\gamma_\mu|q]\gamma^\mu = -\frac{1}{2}\overline{u}_-(\vec{p})\gamma_\mu u_-(\vec{q})\gamma^\mu$$

$$-\frac{1}{2}[p|\gamma_\mu|q\rangle\gamma^\mu = -\frac{1}{2}\overline{u}_+(\vec{p})\gamma_\mu u_+(\vec{q})\gamma^\mu$$

Now we use 50.6, 50.8, 50.10, 50.13, and 50.14 to turn these spinors into twistors:

$$-\frac{1}{2}\langle p|\gamma_\mu|q]\gamma^\mu = -\frac{1}{2}\begin{pmatrix} 0 & \phi^*_{\dot{a}}(\vec{p}) \end{pmatrix}\begin{pmatrix} 0 & \sigma^{\dot{a}a}_\mu \\ \overline{\sigma}^{\dot{a}a}_\mu & 0 \end{pmatrix}\begin{pmatrix} \phi_a(\vec{q}) \\ 0 \end{pmatrix}\gamma^\mu$$

$$-\frac{1}{2}[p|\gamma_\mu|q\rangle\gamma^\mu = -\frac{1}{2}\begin{pmatrix} \phi^a(\vec{p}) & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^{\dot{a}a}_\mu \\ \overline{\sigma}^{\dot{a}a}_\mu & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \phi^{*\dot{a}}(\vec{q}) \end{pmatrix}\gamma^\mu$$

Now we multiply:

$$-\frac{1}{2}\langle p|\gamma_\mu|q]\gamma^\mu = -\frac{1}{2}\phi^*_{\dot{a}}(\vec{p})\overline{\sigma}^{\dot{a}a}_\mu\phi_a(\vec{q})\gamma^\mu$$

$$-\frac{1}{2}[p|\gamma_\mu|q\rangle\gamma^\mu = -\frac{1}{2}\phi^a(\vec{p})\sigma_{\mu a\dot{a}}\phi^{*\dot{a}}(\vec{q})\gamma^\mu$$

These twistors commute, so we can write this as:

$$-\frac{1}{2}\langle p|\gamma_\mu|q]\gamma^\mu = -\frac{1}{2}\phi^*_{\dot{a}}(\vec{p})\phi_a(\vec{q})\begin{pmatrix} 0 & \overline{\sigma}^{\dot{a}a}_\mu\sigma^\mu_{b\dot{b}} \\ \overline{\sigma}^{\dot{a}a}_\mu\overline{\sigma}^{\mu\dot{c}c} & 0 \end{pmatrix}$$

$$-\frac{1}{2}[p|\gamma_\mu|q\rangle\gamma^\mu = -\frac{1}{2}\phi^a(\vec{p})\phi^{*\dot{a}}(\vec{q})\begin{pmatrix} 0 & \sigma_{\mu a\dot{a}}\sigma^\mu_{b\dot{b}} \\ \sigma_{\mu a\dot{a}}\overline{\sigma}^{\mu\dot{c}c} & 0 \end{pmatrix}$$

Now we use 35.19 and 35.4, using the spinor indices to raise and lower the indexes. The result is:

$$-\frac{1}{2}\langle p|\gamma_\mu|q]\gamma^\mu = \begin{pmatrix} 0 & \phi^*_{\dot{b}}(\vec{p})\phi_b(\vec{q}) \\ \phi^{*\dot{c}}(\vec{p})\phi^c(\vec{q}) & 0 \end{pmatrix}$$

$$-\frac{1}{2}[p|\gamma_\mu|q\rangle\gamma^\mu = \begin{pmatrix} 0 & \phi_b(\vec{p})\phi^*_{\dot{b}}(\vec{q}) \\ \phi^c(\vec{p})\phi^{*\dot{c}}(\vec{q}) & 0 \end{pmatrix}$$

Now we can split up this result into spinors:

$$-\frac{1}{2}\langle p|\gamma_\mu|q]\gamma^\mu = \begin{pmatrix} \phi_b(\vec{q}) \\ 0 \end{pmatrix}\begin{pmatrix} 0 & \phi^*_{\dot{b}}(\vec{p}) \end{pmatrix} + \begin{pmatrix} 0 \\ \phi^{*\dot{c}}(\vec{p}) \end{pmatrix}\begin{pmatrix} \phi^c(\vec{q}) & 0 \end{pmatrix}$$

$$-\frac{1}{2}[p|\gamma_\mu|q\rangle\gamma^\mu = \begin{pmatrix} 0 \\ \phi^{*\dot{c}}(\vec{q}) \end{pmatrix}\begin{pmatrix} \phi^c(\vec{p}) & 0 \end{pmatrix} + \begin{pmatrix} \phi_b(\vec{p}) \\ 0 \end{pmatrix}\begin{pmatrix} 0 & \phi^*_{\dot{b}}(\vec{q}) \end{pmatrix}$$

Now we use 50.8, 50.10, 50.13 and 50.14 again:

$$-\frac{1}{2}\langle p|\gamma_\mu|q]\gamma^\mu = u_-(\vec{q})\overline{u}_-(\vec{p}) + u_+(\vec{p})\overline{u}_+(\vec{q})$$

$$-\frac{1}{2}[p|\gamma_\mu|q\rangle\gamma^\mu = u_+(\vec{q})\overline{u}_+(\vec{p}) + u_-(\vec{p})\overline{u}_-(\vec{q})$$

And finally, 50.33:

$$-\frac{1}{2}\langle p|\gamma_\mu|q]\gamma^\mu = |q]\langle p| + |p\rangle[q|$$

$$-\frac{1}{2}[p|\gamma_\mu|q\rangle\gamma^\mu = |q\rangle[p| + |p]\langle q|$$

as expected.

Finally, it is trivial to take this last equation between $\langle r|$ and $|s]$ as indicated:

$$-\frac{1}{2}\langle r|[p|\gamma_\mu|q\rangle\gamma^\mu|s] = \langle r||q\rangle[p||s] + \langle r||p]\langle q||s]$$

On the left hand side, $[p|\gamma_\mu|q\rangle$ is a constant, so we bring it to the front. On the right side, we use 50.34:

$$-\frac{1}{2}[p|\gamma_\mu|q\rangle\langle r|\gamma^\mu|s] = \langle rq\rangle[ps]$$

Now using 50.21, we have:

$$[p|\gamma_\mu|q\rangle\langle r|\gamma^\mu|s] = 2\langle qr\rangle[ps]$$

as expected.