# Srednicki Chapter 49

QFT Problems & Solutions

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**Srednicki 49.1. Let $\Psi$ be a Dirac field (representing the electron and positron), $X$ be a Majorana field (representing the *photino*, the hypothetical supersymmetric partner of the photon, with mass $m_{\tilde{\gamma}}$), and $E_L$ and $E_R$ be two different complex scalar fields (representing the two *selectrons*, the hypothetical supersymmetric partners of the left-handed electron and the right-handed electron, with masses $M_L$ and $M_R$; note that the subscripts L and R are just part of their names and do not signify anything about their Lorentz transformation properties). They interact via**

$$\mathcal{L}_1 = \sqrt{2}eE_L^\dagger \overline{X} P_L \Psi + \sqrt{2}eE_R^\dagger \overline{X} P_R \Psi + \text{h.c.}$$

**where $\alpha = e^2/4\pi \approx 1/137$ is the fine-structure constant, and $P_{L,R} = \frac{1}{2}(1 \mp \gamma_5)$.**

**(a) Write down the Hermitian Conjugate term explicitely.**

Of course, $(AB)^\dagger = B^\dagger A^\dagger$, so the Hermitian conjugate term is:

$$\text{h.c.} = \sqrt{2}e\Psi^\dagger P_L^\dagger \overline{X}^\dagger E_L + \sqrt{2}e\Psi^\dagger P_R^\dagger \overline{X}^\dagger E_R$$

Using 37.17 and $\beta^2 = I$:

$$\text{h.c.} = \sqrt{2}e\overline{\Psi}\beta P_L^\dagger \beta X E_L + \sqrt{2}e\overline{\Psi}\beta P_R^\dagger \beta X E_R$$

$E$ is a scalar, which commutes, so we'll just move it bak to the front of the terms. Using 38.14, $\beta P_{L,R}\beta = \overline{P_{L,R}}$. Now recall from 38.15 that $\overline{i\gamma_5} = i\gamma_5$: this implies that $\overline{P_{L,R}} = P_{R,L}$. Thus:

$$\text{h.c.} = \sqrt{2}eE_L\overline{\Psi}P_R X + \sqrt{2}eE_R\overline{\Psi}P_L X$$

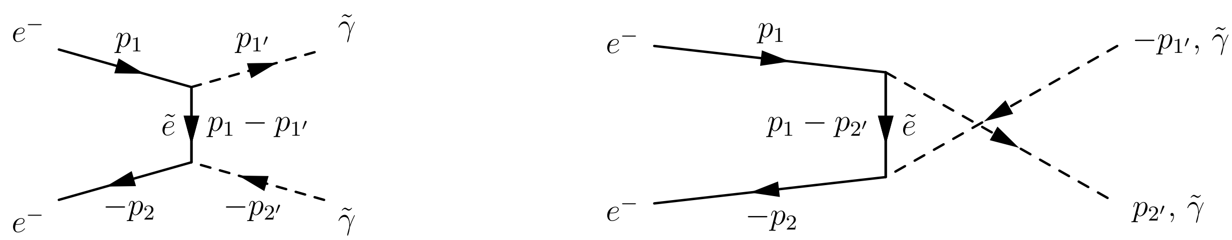
**(b) Find the tree-level scattering amplitude for $e^+e^- \to \tilde{\gamma}\tilde{\gamma}$.**

Let's be a little bit clear with our notation. We have:

- $\tilde{e}$, "selectrons", $E_L$ or $E_R$, are <u>complex scalar fields</u>.
- $\tilde{\gamma}$, "photinos", $X$, are <u>Majorana fields</u>.
- $e^-$ and $e^+$, "electrons and positrons", $\Psi$ or $\overline{\Psi}$, are <u>Dirac fields</u>.

Now we have this monster 4-term interaction Lagrangian, but all of them join a selectron, a photino, and an electron/positron.

At tree level (no loops), we can draw these diagrams:




A few comments about these diagrams:

- As is customary, we write all particles as particles, not antiparticles. Antiparticles have backwards arrows.

- Notice that to keep the labeling the same in the second diagram as in the first, we have to introduce one cross. This means that there is a relative minus sign.

- The direction of the arrows and sign of the four-momentum is fixed by the Feynman rules. It is very important to get these signs right! (The internal scalar can have the arrow in either direction.) For the Dirac particles ($e^-$ and $e^+$), we have to refer back to the Feynman rules presented in chapter 45.

- There are two copies of each diagram, one for $\tilde{e_1}$ and one for $\tilde{e_2}$.

- The diagram on the left is t-channel; the diagram on the right is u-channel.

Now we're ready to assess the values of these diagrams. Using the full Lagrangian, we see that there are four possible vertices. These are:

- electron, selectron 1, photino: $i\sqrt{2}eP_L$

- electron, selectron 2, photino: $i\sqrt{2}eP_R$

- positron, selectron 1, photino: $i\sqrt{2}eP_L$

- positron, selectron 2, photino: $i\sqrt{2}eP_R$

Following the Feynman Rules, we trace the fermionic lines backwards, writing down the factors we encounter. We alternate between following the rules for Majorana fields (when dealing with photinos) and those for Dirac fields (when dealing with electrons or positrons). The prescriptions for dealing with the internal scalar is the same. The vertex factors are different from those in either set of Feynman Rules; we specifed those above. Then:

$$i\mathcal{T} = \overline{v}_{s_2}(\vec{p}_2)\left[\sqrt{2}eP_R\right]v_{s_{2'}}(\vec{p}_{2'})\frac{(-i)}{(p_1 - p_{1'})^2 + M_L^2 - i\varepsilon}\overline{u}_{s_{1'}}(\vec{p}_{1'})\left[\sqrt{2}eP_L\right]u_{s_1}(\vec{p}_1)$$

$$+\overline{v}_{s_2}(\vec{p}_2)\left[\sqrt{2}eP_L\right]v_{s_{2'}}(\vec{p}_{2'})\frac{(-i)}{(p_1-p_{1'})^2+M_R^2-i\varepsilon}\overline{u}_{s_{1'}}(\vec{p}_{1'})\left[\sqrt{2}eP_R\right]u_{s_1}(\vec{p}_1)$$

$$-\overline{v}_{s_2}(\vec{p}_2)\left[\sqrt{2}eP_R\right]v_{s_{1'}}(\vec{p}_{1'})\frac{(-i)}{(p_1-p_{2'})^2+M_L^2-i\varepsilon}\overline{u}_{s_{2'}}(\vec{p}_{2'})\left[\sqrt{2}eP_L\right]u_{s_1}(\vec{p}_1)$$

$$-\overline{v}_{s_2}(\vec{p}_2)\left[\sqrt{2}eP_L\right]v_{s_{1'}}(\vec{p}_{1'})\frac{(-i)}{(p_1-p_{2'})^2+M_R^2-i\varepsilon}\overline{u}_{s_{2'}}(\vec{p}_{2'})\left[\sqrt{2}eP_R\right]u_{s_1}(\vec{p}_1)$$

Now we need to simplify this mess:

$$\mathcal{T}=2e^2\left\{\frac{\overline{v}_{s_2}(\vec{p}_2)P_Rv_{s_{2'}}(\vec{p}_{2'})\overline{u}_{s_{1'}}(\vec{p}_{1'})P_Lu_{s_1}(\vec{p}_1)}{(p_1-p_{1'})^2+M_L^2-i\varepsilon}+\frac{\overline{v}_{s_2}(\vec{p}_2)P_Lv_{s_{2'}}(\vec{p}_{2'})\overline{u}_{s_{1'}}(\vec{p}_{1'})P_Ru_{s_1}(\vec{p}_1)}{(p_1-p_{1'})^2+M_R^2-i\varepsilon}\right.$$
$$\left.-\frac{\overline{v}_{s_2}(\vec{p}_2)P_Rv_{s_{1'}}(\vec{p}_{1'})\overline{u}_{s_{2'}}(\vec{p}_{2'})P_Lu_{s_1}(\vec{p}_1)}{(p_1-p_{2'})^2+M_L^2-i\varepsilon}-\frac{\overline{v}_{s_2}(\vec{p}_2)P_Lv_{s_{1'}}(\vec{p}_{1'})\overline{u}_{s_{2'}}(\vec{p}_{2'})P_Ru_{s_1}(\vec{p}_1)}{(p_1-p_{2'})^2+M_R^2-i\varepsilon}\right\}$$

To make this a little easier for ourselves, let's reduce to one type of spinor. This is of course not usually possible, but thanks to the projection matrices, we note that:

$$\overline{v}P_{L,R}v'=(\overline{v}P_{L,R}v')^T=v'^TP_{L,R}^T\overline{v}^T=\overline{u}'\mathcal{C}^{-1}P_{L,R}^T\mathcal{C}^{-1}u=-\overline{u}'\mathcal{C}^{-1}P_{L,R}^T\mathcal{C}u=-\overline{u}'P_{L,R}u$$

where the first equality is possible because $\overline{v}P_{L,R}v'$ is a scalar, the second equality just uses $(AB)^T=B^TA^T$, the third equality uses 38.37 and 38.34, and the fourth equality uses 38.34. For the final equality, we recall that the projection matrices are explicitely written in 36.44, and so it is easy to use 38.33 to verify that $\mathcal{C}^{-1}P_{L,R}^T\mathcal{C}=P_{L,R}$.

This allows us to write $\mathcal{T}$ in terms of $u$ spinors only:

$$\mathcal{T}=-2e^2\left\{\frac{\overline{u}_{s_{2'}}(\vec{p}_{2'})P_Ru_{s_2}(\vec{p}_2)\overline{u}_{s_{1'}}(\vec{p}_{1'})P_Lu_{s_1}(\vec{p}_1)}{(p_1-p_{1'})^2+M_L^2-i\varepsilon}+\frac{\overline{u}_{s_{2'}}(\vec{p}_{2'})P_Lu_{s_2}(\vec{p}_2)\overline{u}_{s_{1'}}(\vec{p}_{1'})P_Ru_{s_1}(\vec{p}_1)}{(p_1-p_{1'})^2+M_R^2-i\varepsilon}\right.$$
$$\left.-\frac{\overline{u}_{s_{1'}}(\vec{p}_{1'})P_Ru_{s_2}(\vec{p}_2)\overline{u}_{s_{2'}}(\vec{p}_{2'})P_Lu_{s_1}(\vec{p}_1)}{(p_1-p_{2'})^2+M_L^2-i\varepsilon}-\frac{\overline{u}_{s_{1'}}(\vec{p}_{1'})P_Lu_{s_2}(\vec{p}_2)\overline{u}_{s_{2'}}(\vec{p}_{2'})P_Ru_{s_1}(\vec{p}_1)}{(p_1-p_{2'})^2+M_R^2-i\varepsilon}\right\}$$

We can also introduce Mandelstam variables into the denominator:

$$\mathcal{T}=-2e^2\left\{\frac{\overline{u}_{s_{2'}}(\vec{p}_{2'})P_Ru_{s_2}(\vec{p}_2)\overline{u}_{s_{1'}}(\vec{p}_{1'})P_Lu_{s_1}(\vec{p}_1)}{-t+M_L^2-i\varepsilon}+\frac{\overline{u}_{s_{2'}}(\vec{p}_{2'})P_Lu_{s_2}(\vec{p}_2)\overline{u}_{s_{1'}}(\vec{p}_{1'})P_Ru_{s_1}(\vec{p}_1)}{-t+M_R^2-i\varepsilon}\right.$$
$$\left.-\frac{\overline{u}_{s_{1'}}(\vec{p}_{1'})P_Ru_{s_2}(\vec{p}_2)\overline{u}_{s_{2'}}(\vec{p}_{2'})P_Lu_{s_1}(\vec{p}_1)}{-u+M_L^2-i\varepsilon}-\frac{\overline{u}_{s_{1'}}(\vec{p}_{1'})P_Lu_{s_2}(\vec{p}_2)\overline{u}_{s_{2'}}(\vec{p}_{2'})P_Ru_{s_1}(\vec{p}_1)}{-u+M_R^2-i\varepsilon}\right\}$$

That's about as much as we can do.

**(c) Compute the spin-averaged differential cross-section for this process in the case that $m_e$ can be neglected, and $|t|,|u|\ll M_L=M_R$. Express it as a function of $s$ and the center-of-mass scattering angle $\theta$.**

*Note: These assumptions are completely reasonable. In most models, the selectron mass ($M_L$ or $M_R$) is very high, several hundred GeV at least. The most fashionable models today* *("natural SUSY models") generally set the selectron mass as high as several TeV.*

With these assumptions, we can simplify our result a little from part (b). In particular, the selectron mass is so high that we can neglect the Mandelstam variable in the denominator; it is then obvious that we can neglect the infinitesimal as well. We also define $M = M_L = M_R$. Finally, this math is going to get a little messy, so let's neglect writing the three-momenta and spins (these are implied by the subscripts):

$$\mathcal{T} = -\frac{2e^2}{M^2}\left[\overline{u}_{2'}P_R u_2\overline{u}_{1'}P_L u_1 + \overline{u}_{2'}P_L u_2\overline{u}_{1'}P_R u_1 - \overline{u}_{1'}P_R u_2\overline{u}_{2'}P_L u_1 - \overline{u}_{1'}P_L u_2\overline{u}_{2'}P_R u_1\right]$$

Now we can bar this. Recall that $\overline{i\gamma_5} = i\gamma_5$, so $\overline{P}_{L,R} = P_{R,L}$. Then:

$$\overline{\mathcal{T}} = -\frac{2e^2}{M^2}\left[\overline{u}_1 P_R u_{1'}\overline{u}_2 P_L u_{2'} + \overline{u}_1 P_L u_{1'}\overline{u}_2 P_R u_{2'} - \overline{u}_1 P_R u_{2'}\overline{u}_2 P_L u_{1'} - \overline{u}_1 P_L u_{2'}\overline{u}_2 P_R u_{1'}\right]$$

Now we can multiply these together. This is where it hits the fan!

$$|\mathcal{T}|^2 = \frac{4e^4}{M^4}\left[\overline{u}_{2'}P_R u_2\overline{u}_{1'}P_L u_1 + \overline{u}_{2'}P_L u_2\overline{u}_{1'}P_R u_1 - \overline{u}_{1'}P_R u_2\overline{u}_{2'}P_L u_1 - \overline{u}_{1'}P_L u_2\overline{u}_{2'}P_R u_1\right]$$

$$\left[\overline{u}_1 P_R u_{1'}\overline{u}_2 P_L u_{2'} + \overline{u}_1 P_L u_{1'}\overline{u}_2 P_R u_{2'} - \overline{u}_1 P_R u_{2'}\overline{u}_2 P_L u_{1'} - \overline{u}_1 P_L u_{2'}\overline{u}_2 P_R u_{1'}\right]$$

Expanding:

$$|\mathcal{T}|^2 = \frac{4e^4}{M^4}\left[\overline{u}_{2'}P_R u_2\overline{u}_{1'}P_L u_1\overline{u}_1 P_R u_{1'}\overline{u}_2 P_L u_{2'} + \overline{u}_{2'}P_R u_2\overline{u}_{1'}P_L u_1\overline{u}_1 P_L u_{1'}\overline{u}_2 P_R u_{2'}\right.$$

$$-\overline{u}_{2'}P_R u_2\overline{u}_{1'}P_L u_1\overline{u}_1 P_R u_{2'}\overline{u}_2 P_L u_{1'} - \overline{u}_{2'}P_R u_2\overline{u}_{1'}P_L u_1\overline{u}_1 P_L u_{2'}\overline{u}_2 P_R u_{1'} + \overline{u}_{2'}P_L u_2\overline{u}_{1'}P_R u_1\overline{u}_1 P_R u_{1'}\overline{u}_2 P_L u_{2'}$$

$$+\overline{u}_{2'}P_L u_2\overline{u}_{1'}P_R u_1\overline{u}_1 P_L u_{1'}\overline{u}_2 P_R u_{2'} - \overline{u}_{2'}P_L u_2\overline{u}_{1'}P_R u_1\overline{u}_1 P_R u_{2'}\overline{u}_2 P_L u_{1'} - \overline{u}_{2'}P_L u_2\overline{u}_{1'}P_R u_1\overline{u}_1 P_L u_{2'}\overline{u}_2 P_R u_{1'}$$

$$-\overline{u}_{1'}P_R u_2\overline{u}_{2'}P_L u_1\overline{u}_1 P_R u_{1'}\overline{u}_2 P_L u_{2'} - \overline{u}_{1'}P_R u_2\overline{u}_{2'}P_L u_1\overline{u}_1 P_L u_{1'}\overline{u}_2 P_R u_{2'} + \overline{u}_{1'}P_R u_2\overline{u}_{2'}P_L u_1\overline{u}_1 P_R u_{2'}\overline{u}_2 P_L u_{1'}$$

$$+\overline{u}_{1'}P_R u_2\overline{u}_{2'}P_L u_1\overline{u}_1 P_L u_{2'}\overline{u}_2 P_R u_{1'} - \overline{u}_{1'}P_L u_2\overline{u}_{2'}P_R u_1\overline{u}_1 P_R u_{1'}\overline{u}_2 P_L u_{2'} - \overline{u}_{1'}P_L u_2\overline{u}_{2'}P_R u_1\overline{u}_1 P_L u_{1'}\overline{u}_2 P_R u_{2'}$$

$$\left.+\overline{u}_{1'}P_L u_2\overline{u}_{2'}P_R u_1\overline{u}_1 P_R u_{2'}\overline{u}_2 P_L u_{1'} + \overline{u}_{1'}P_L u_2\overline{u}_{2'}P_R u_1\overline{u}_1 P_L u_{2'}\overline{u}_2 P_R u_{1'}\right]$$

Now we want to put similar terms near one another. A barred spinor, projection matrix, and spinor form a scalar, so we can move these around as we like. Then:

$$|\mathcal{T}|^2 = \frac{4e^4}{M^4}\left[\overline{u}_{2'}P_R u_2\overline{u}_2 P_L u_{2'}\overline{u}_{1'}P_L u_1\overline{u}_1 P_R u_{1'} + \overline{u}_{2'}P_R u_2\overline{u}_2 P_R u_{2'}\overline{u}_{1'}P_L u_1\overline{u}_1 P_L u_{1'}\right.$$

$$-\overline{u}_{2'}P_R u_2\overline{u}_2 P_L u_{1'}\overline{u}_{1'}P_L u_1\overline{u}_1 P_R u_{2'} - \overline{u}_{2'}P_R u_2\overline{u}_2 P_R u_{1'}\overline{u}_{1'}P_L u_1\overline{u}_1 P_L u_{2'} + \overline{u}_{2'}P_L u_2\overline{u}_2 P_L u_{2'}\overline{u}_{1'}P_R u_1\overline{u}_1 P_R u_{1'}$$

$$+\overline{u}_{2'}P_L u_2\overline{u}_2 P_R u_{2'}\overline{u}_{1'}P_R u_1\overline{u}_1 P_L u_{1'} - \overline{u}_{2'}P_L u_2\overline{u}_2 P_L u_{1'}\overline{u}_{1'}P_R u_1\overline{u}_1 P_R u_{2'} - \overline{u}_{2'}P_L u_2\overline{u}_2 P_R u_{1'}\overline{u}_{1'}P_R u_1\overline{u}_1 P_L u_{2'}$$

$$-\overline{u}_{1'}P_R u_2\overline{u}_2 P_L u_{2'}\overline{u}_{2'}P_L u_1\overline{u}_1 P_R u_{1'} - \overline{u}_{1'}P_R u_2\overline{u}_2 P_R u_{2'}\overline{u}_{2'}P_L u_1\overline{u}_1 P_L u_{1'} + \overline{u}_{1'}P_R u_2\overline{u}_2 P_L u_{1'}\overline{u}_{2'}P_L u_1\overline{u}_1 P_R u_{2'}$$

$$+\overline{u}_{1'}P_R u_2\overline{u}_2 P_R u_{1'}\overline{u}_{2'}P_L u_1\overline{u}_1 P_L u_{2'} - \overline{u}_{1'}P_L u_2\overline{u}_2 P_L u_{2'}\overline{u}_{2'}P_R u_1\overline{u}_1 P_R u_{1'} - \overline{u}_{1'}P_L u_2\overline{u}_2 P_R u_{2'}\overline{u}_{2'}P_R u_1\overline{u}_1 P_L u_{1'}$$

$$\left.+\overline{u}_{1'}P_L u_2\overline{u}_2 P_L u_{1'}\overline{u}_{2'}P_R u_1\overline{u}_1 P_R u_{2'} + \overline{u}_{1'}P_L u_2\overline{u}_2 P_R u_{1'}\overline{u}_{2'}P_R u_1\overline{u}_1 P_L u_{2'}\right]$$

Now we use our usual trick of writing in index notation, creating a trace, then dropping the index notation. In some cases we choose to make two traces, in other terms we choose to make just one – whatever is needed to keep similar terms next to one another (so that we can use the Dirac equation).

$$|\mathcal{T}|^2 = \frac{4e^4}{M^4}\{\ \mathrm{Tr}\,[\overline{u}_{2'}P_R u_2\overline{u}_2 P_L u_{2'}]\ \ \mathrm{Tr}\,[\overline{u}_{1'}P_L u_1\overline{u}_1 P_R u_{1'}] +\ \mathrm{Tr}\,[\overline{u}_{2'}P_R u_2\overline{u}_2 P_R u_{2'}]\ \ \mathrm{Tr}\,[\overline{u}_{1'}P_L u_1\overline{u}_1 P_L u_{1'}]$$
$$-\ \mathrm{Tr}\,[\overline{u}_{2'}P_R u_2\overline{u}_2 P_L u_{1'}\overline{u}_{1'}P_L u_1\overline{u}_1 P_R u_{2'}] -\ \mathrm{Tr}\,[\overline{u}_{2'}P_R u_2\overline{u}_2 P_R u_{1'}\overline{u}_{1'}P_L u_1\overline{u}_1 P_L u_{2'}]$$
$$+\ \mathrm{Tr}\,[\overline{u}_{2'}P_L u_2\overline{u}_2 P_L u_{2'}]\ \ \mathrm{Tr}\,[\overline{u}_{1'}P_R u_1\overline{u}_1 P_R u_{1'}] +\ \mathrm{Tr}\,[\overline{u}_{2'}P_L u_2\overline{u}_2 P_R u_{2'}]\ \ \mathrm{Tr}\,[\overline{u}_{1'}P_R u_1\overline{u}_1 P_L u_{1'}]$$
$$-\ \mathrm{Tr}\,[\overline{u}_{2'}P_L u_2\overline{u}_2 P_L u_{1'}\overline{u}_{1'}P_R u_1\overline{u}_1 P_R u_{2'}] -\ \mathrm{Tr}\,[\overline{u}_{2'}P_L u_2\overline{u}_2 P_R u_{1'}\overline{u}_{1'}P_R u_1\overline{u}_1 P_L u_{2'}]$$
$$-\ \mathrm{Tr}\,[\overline{u}_{1'}P_R u_2\overline{u}_2 P_L u_{2'}\overline{u}_{2'}P_L u_1\overline{u}_1 P_R u_{1'}] -\ \mathrm{Tr}\,[\overline{u}_{1'}P_R u_2\overline{u}_2 P_R u_{2'}\overline{u}_{2'}P_L u_1\overline{u}_1 P_L u_{1'}]$$
$$+\ \mathrm{Tr}\,[\overline{u}_{1'}P_R u_2\overline{u}_2 P_L u_{1'}]\ \ \mathrm{Tr}\,[\overline{u}_{2'}P_L u_1\overline{u}_1 P_R u_{2'}] +\ \mathrm{Tr}\,[\overline{u}_{1'}P_R u_2\overline{u}_2 P_R u_{1'}]\ \ \mathrm{Tr}\,[\overline{u}_{2'}P_L u_1\overline{u}_1 P_L u_{2'}]$$
$$-\ \mathrm{Tr}\,[\overline{u}_{1'}P_L u_2\overline{u}_2 P_L u_{2'}\overline{u}_{2'}P_R u_1\overline{u}_1 P_R u_{1'}] -\ \mathrm{Tr}\,[\overline{u}_{1'}P_L u_2\overline{u}_2 P_R u_{2'}\overline{u}_{2'}P_R u_1\overline{u}_1 P_L u_{1'}]$$
$$+\ \mathrm{Tr}\,[\overline{u}_{1'}P_L u_2\overline{u}_2 P_L u_{1'}]\ \ \mathrm{Tr}\,[\overline{u}_{2'}P_R u_1\overline{u}_1 P_R u_{2'}] +\ \mathrm{Tr}\,[\overline{u}_{1'}P_L u_2\overline{u}_2 P_R u_{1'}]\ \ \mathrm{Tr}\,[\overline{u}_{2'}P_R u_1\overline{u}_1 P_L u_{2'}]\}$$

Now we're ready to average over the inital spins and sum over the final spins:

$$\langle|\mathcal{T}|^2\rangle = \frac{e^4}{M^4}\sum_{\text{spins}}\{\ \mathrm{Tr}\,[\overline{u}_{2'}P_R u_2\overline{u}_2 P_L u_{2'}]\ \ \mathrm{Tr}\,[\overline{u}_{1'}P_L u_1\overline{u}_1 P_R u_{1'}] +\ \mathrm{Tr}\,[\overline{u}_{2'}P_R u_2\overline{u}_2 P_R u_{2'}]\ \ \mathrm{Tr}\,[\overline{u}_{1'}P_L u_1\overline{u}_1 P_L u_{1'}]$$
$$-\ \mathrm{Tr}\,[\overline{u}_{2'}P_R u_2\overline{u}_2 P_L u_{1'}\overline{u}_{1'}P_L u_1\overline{u}_1 P_R u_{2'}] -\ \mathrm{Tr}\,[\overline{u}_{2'}P_R u_2\overline{u}_2 P_R u_{1'}\overline{u}_{1'}P_L u_1\overline{u}_1 P_L u_{2'}]$$
$$+\ \mathrm{Tr}\,[\overline{u}_{2'}P_L u_2\overline{u}_2 P_L u_{2'}]\ \ \mathrm{Tr}\,[\overline{u}_{1'}P_R u_1\overline{u}_1 P_R u_{1'}] +\ \mathrm{Tr}\,[\overline{u}_{2'}P_L u_2\overline{u}_2 P_R u_{2'}]\ \ \mathrm{Tr}\,[\overline{u}_{1'}P_R u_1\overline{u}_1 P_L u_{1'}]$$
$$-\ \mathrm{Tr}\,[\overline{u}_{2'}P_L u_2\overline{u}_2 P_L u_{1'}\overline{u}_{1'}P_R u_1\overline{u}_1 P_R u_{2'}] -\ \mathrm{Tr}\,[\overline{u}_{2'}P_L u_2\overline{u}_2 P_R u_{1'}\overline{u}_{1'}P_R u_1\overline{u}_1 P_L u_{2'}]$$
$$-\ \mathrm{Tr}\,[\overline{u}_{1'}P_R u_2\overline{u}_2 P_L u_{2'}\overline{u}_{2'}P_L u_1\overline{u}_1 P_R u_{1'}] -\ \mathrm{Tr}\,[\overline{u}_{1'}P_R u_2\overline{u}_2 P_R u_{2'}\overline{u}_{2'}P_L u_1\overline{u}_1 P_L u_{1'}]$$
$$+\ \mathrm{Tr}\,[\overline{u}_{1'}P_R u_2\overline{u}_2 P_L u_{1'}]\ \ \mathrm{Tr}\,[\overline{u}_{2'}P_L u_1\overline{u}_1 P_R u_{2'}] +\ \mathrm{Tr}\,[\overline{u}_{1'}P_R u_2\overline{u}_2 P_R u_{1'}]\ \ \mathrm{Tr}\,[\overline{u}_{2'}P_L u_1\overline{u}_1 P_L u_{2'}]$$
$$-\ \mathrm{Tr}\,[\overline{u}_{1'}P_L u_2\overline{u}_2 P_L u_{2'}\overline{u}_{2'}P_R u_1\overline{u}_1 P_R u_{1'}] -\ \mathrm{Tr}\,[\overline{u}_{1'}P_L u_2\overline{u}_2 P_R u_{2'}\overline{u}_{2'}P_R u_1\overline{u}_1 P_L u_{1'}]$$
$$+\ \mathrm{Tr}\,[\overline{u}_{1'}P_L u_2\overline{u}_2 P_L u_{1'}]\ \ \mathrm{Tr}\,[\overline{u}_{2'}P_R u_1\overline{u}_1 P_R u_{2'}] +\ \mathrm{Tr}\,[\overline{u}_{1'}P_L u_2\overline{u}_2 P_R u_{1'}]\ \ \mathrm{Tr}\,[\overline{u}_{2'}P_R u_1\overline{u}_1 P_L u_{2'}]\}$$

Now to use the Dirac Equations. Let's start with $\sum_{s_1,s_2} = -\not{p}_i$, where we neglect the electron mass. Then:

$$\langle|\mathcal{T}|^2\rangle = \frac{e^4}{M^4}\sum_{s_{1'},s_{2'}}\Big\{\ \mathrm{Tr}\left[\overline{u}_{2'}P_R(-\not{p}_2)P_L u_{2'}\right]\ \ \mathrm{Tr}\left[\overline{u}_{1'}P_L(-\not{p}_1)P_R u_{1'}\right]$$
$$+\ \mathrm{Tr}\left[\overline{u}_{2'}P_R(-\not{p}_2)P_R u_{2'}\right]\ \ \mathrm{Tr}\left[\overline{u}_{1'}P_L(-\not{p}_1)P_L u_{1'}\right]$$
$$-\ \mathrm{Tr}\left[\overline{u}_{2'}P_R(-\not{p}_2)P_L u_{1'}\overline{u}_{1'}P_L(-\not{p}_1)P_R u_{2'}\right] -\ \mathrm{Tr}\left[\overline{u}_{2'}P_R(-\not{p}_2)P_R u_{1'}\overline{u}_{1'}P_L(-\not{p}_1)P_L u_{2'}\right]$$
$$+\ \mathrm{Tr}\left[\overline{u}_{2'}P_L(-\not{p}_2)P_L u_{2'}\right]\ \ \mathrm{Tr}\left[\overline{u}_{1'}P_R(-\not{p}_1)P_R u_{1'}\right] +\ \mathrm{Tr}\left[\overline{u}_{2'}P_L(-\not{p}_2)P_R u_{2'}\right]\ \ \mathrm{Tr}\left[\overline{u}_{1'}P_R(-\not{p}_1)P_L u_{1'}\right]$$
$$-\ \mathrm{Tr}\left[\overline{u}_{2'}P_L(-\not{p}_2)P_L u_{1'}\overline{u}_{1'}P_R(-\not{p}_1)P_R u_{2'}\right] -\ \mathrm{Tr}\left[\overline{u}_{2'}P_L(-\not{p}_2)P_R u_{1'}\overline{u}_{1'}P_R(-\not{p}_1)P_L u_{2'}\right]$$

$$- \text{Tr}\left[\overline{u}_{1'} P_R(-\not{p}_2) P_L u_{2'}\overline{u}_{2'} P_L(-\not{p}_1) P_R u_{1'}\right] - \text{Tr}\left[\overline{u}_{1'} P_R(-\not{p}_2) P_R u_{2'}\overline{u}_{2'} P_L(-\not{p}_1) P_L u_{1'}\right]$$

$$+ \text{Tr}\left[\overline{u}_{1'} P_R(-\not{p}_2) P_L u_{1'}\right] \text{Tr}\left[\overline{u}_{2'} P_L(-\not{p}_1) P_R u_{2'}\right] + \text{Tr}\left[\overline{u}_{1'} P_R(-\not{p}_2) P_R u_{1'}\right] \text{Tr}\left[\overline{u}_{2'} P_L(-\not{p}_1) P_L u_{2'}\right]$$

$$- \text{Tr}\left[\overline{u}_{1'} P_L(-\not{p}_2) P_L u_{2'}\overline{u}_{2'} P_R(-\not{p}_1) P_R u_{1'}\right] - \text{Tr}\left[\overline{u}_{1'} P_L(-\not{p}_2) P_R u_{2'}\overline{u}_{2'} P_R(-\not{p}_1) P_L u_{1'}\right]$$

$$+ \text{Tr}\left[\overline{u}_{1'} P_L(-\not{p}_2) P_L u_{1'}\right] \text{Tr}\left[\overline{u}_{2'} P_R(-\not{p}_1) P_R u_{2'}\right] + \text{Tr}\left[\overline{u}_{1'} P_L(-\not{p}_2) P_R u_{1'}\right] \text{Tr}\left[\overline{u}_{2'} P_R(-\not{p}_1) P_L u_{2'}\right]\bigg\}$$

Now we have 16 terms! Fortunately, we're about to cancel half of them. Notice that $P_{L,R}\not{p}_i = \not{p}_i P_{R,L}$, due to anticommutation of $\gamma_5$. Further, remember that these are projection operators, and so $P_L P_R = P_R P_L = 0$. Thus, if we have two copies of the same projection operator separated by a slash, the term will vanish. This gives:

$$\langle|\mathcal{T}|^2\rangle = \frac{e^4}{M^4}\sum_{s_{1'},s_{2'}}\bigg\{ \text{Tr}\left[\overline{u}_{2'} P_R(-\not{p}_2) P_L u_{2'}\right] \text{Tr}\left[u_{1'}\overline{u}_{1'} P_L(-\not{p}_1) P_R\right]$$

$$- \text{Tr}\left[\overline{u}_{2'} P_R(-\not{p}_2) P_L u_{1'}\overline{u}_{1'} P_L(-\not{p}_1) P_R u_{2'}\right] + \text{Tr}\left[\overline{u}_{2'} P_L(-\not{p}_2) P_R u_{2'}\right] \text{Tr}\left[u_{1'}\overline{u}_{1'} P_R(-\not{p}_1) P_L\right]$$

$$- \text{Tr}\left[\overline{u}_{2'} P_L(-\not{p}_2) P_R u_{1'}\overline{u}_{1'} P_R(-\not{p}_1) P_L u_{2'}\right] - \text{Tr}\left[u_{1'}\overline{u}_{1'} P_R(-\not{p}_2) P_L u_{2'}\overline{u}_{2'} P_L(-\not{p}_1) P_R\right]$$

$$+ \text{Tr}\left[\overline{u}_{1'} P_R(-\not{p}_2) P_L u_{1'}\right] \text{Tr}\left[\overline{u}_{2'} P_L(-\not{p}_1) P_R u_{2'}\right] - \text{Tr}\left[u_{1'}\overline{u}_{1'} P_L(-\not{p}_2) P_R u_{2'}\overline{u}_{2'} P_R(-\not{p}_1) P_L\right]$$

$$+ \text{Tr}\left[\overline{u}_{1'} P_L(-\not{p}_2) P_R u_{1'}\right] \text{Tr}\left[\overline{u}_{2'} P_R(-\not{p}_1) P_L u_{2'}\right]\bigg\}$$

Now let's take the remaining sums, using the Dirac Equation ($\sum_{s_{i'}} u_{i'}\overline{u}_{i'} = -\not{p}_{i'} + m$, where $m$ is the photino mass):

$$\langle|\mathcal{T}|^2\rangle = \frac{e^4}{M^4}\bigg\{ \text{Tr}\left[(-\not{p}_{2'} + m) P_R(-\not{p}_2) P_L\right] \text{Tr}\left[(-\not{p}_{1'} + m) P_L(-\not{p}_1) P_R\right]$$

$$- \text{Tr}\left[(-\not{p}_{2'} + m) P_R(-\not{p}_2) P_L(-\not{p}_{1'} + m) P_L(-\not{p}_1) P_R\right]$$

$$+ \text{Tr}\left[(-\not{p}_{2'} + m) P_L(-\not{p}_2) P_R\right] \text{Tr}\left[(-\not{p}_{1'} + m) P_R(-\not{p}_1) P_L\right]$$

$$- \text{Tr}\left[(-\not{p}_{2'} + m) P_L(-\not{p}_2) P_R(-\not{p}_{1'} + m) P_R(-\not{p}_1) P_L\right]$$

$$- \text{Tr}\left[(-\not{p}_{1'} + m) P_R(-\not{p}_2) P_L(-\not{p}_{2'} + m) P_L(-\not{p}_1) P_R\right]$$

$$+ \text{Tr}\left[(-\not{p}_{1'} + m) P_R(-\not{p}_2) P_L\right] \text{Tr}\left[(-\not{p}_{2'} + m) P_L(-\not{p}_1) P_R\right]$$

$$- \text{Tr}\left[(-\not{p}_{1'} + m) P_L(-\not{p}_2) P_R(-\not{p}_{2'} + m) P_R(-\not{p}_1) P_L\right]$$

$$+ \text{Tr}\left[(-\not{p}_{1'} + m) P_L(-\not{p}_2) P_R\right] \text{Tr}\left[(-\not{p}_{2'} + m) P_R(-\not{p}_1) P_L\right]\bigg\}$$

Some of these slashes vanish, for the same reason as before (sandwiched by the same projection operator):

$$\langle|\mathcal{T}|^2\rangle = \frac{e^4}{M^4}\Big\{ \text{ Tr}\Big[(-\not{p}_{2'}+m)P_R(-\not{p}_2)P_L\Big] \text{ Tr}\Big[(-\not{p}_{1'}+m)P_L(-\not{p}_1)P_R\Big]$$

$$- \text{ Tr}\Big[(m)P_R(-\not{p}_2)P_L(m)P_L(-\not{p}_1)P_R\Big]$$

$$+ \text{ Tr}\Big[(-\not{p}_{2'}+m)P_L(-\not{p}_2)P_R\Big] \text{ Tr}\Big[(-\not{p}_{1'}+m)P_R(-\not{p}_1)P_L\Big]$$

$$- \text{ Tr}\Big[(m)P_L(-\not{p}_2)P_R(m)P_R(-\not{p}_1)P_L\Big] - \text{ Tr}\Big[(m)P_R(-\not{p}_2)P_L(m)P_L(-\not{p}_1)P_R\Big]$$

$$+ \text{ Tr}\Big[(-\not{p}_{1'}+m)P_R(-\not{p}_2)P_L\Big] \text{ Tr}\Big[(-\not{p}_{2'}+m)P_L(-\not{p}_1)P_R\Big]$$

$$- \text{ Tr}\Big[(m)P_L(-\not{p}_2)P_R(m)P_R(-\not{p}_1)P_L\Big]$$

$$+ \text{ Tr}\Big[(-\not{p}_{1'}+m)P_L(-\not{p}_2)P_R\Big] \text{ Tr}\Big[(-\not{p}_{2'}+m)P_R(-\not{p}_1)P_L\Big]\Big\}$$

Next, we recall that all projection operators follow $P^2 = P$. Thus:

$$\langle|\mathcal{T}|^2\rangle = \frac{e^4}{M^4}\Big\{ \text{ Tr}\Big[(-\not{p}_{2'}+m)(\not{p}_2)P_L\Big] \text{ Tr}\Big[(-\not{p}_{1'}+m)(\not{p}_1)P_R\Big] - m^2 \text{ Tr}\Big[(\not{p}_2)(\not{p}_1)P_R\Big]$$

$$+ \text{ Tr}\Big[(-\not{p}_{2'}+m)(\not{p}_2)P_R\Big] \text{ Tr}\Big[(-\not{p}_{1'}+m)(\not{p}_1)P_L\Big] - m^2 \text{ Tr}\Big[(\not{p}_2)(\not{p}_1)P_L\Big] - m^2 \text{ Tr}\Big[(\not{p}_2)(\not{p}_1)P_R\Big]$$

$$+ \text{ Tr}\Big[(-\not{p}_{1'}+m)(\not{p}_2)P_L\Big] \text{ Tr}\Big[(-\not{p}_{2'}+m)(\not{p}_1)P_R\Big] - m^2 \text{ Tr}\Big[(\not{p}_2)(\not{p}_1)P_L\Big]$$

$$+ \text{ Tr}\Big[(-\not{p}_{1'}+m)(\not{p}_2)P_R\Big] \text{ Tr}\Big[(-\not{p}_{2'}+m)(\not{p}_1)P_L\Big]\Big\}$$

Four of these terms combine to two. We also distribute a little bit:

$$\langle|\mathcal{T}|^2\rangle = \frac{e^4}{M^4}\Big\{ \text{ Tr}\Big[(-\not{p}_{2'}\not{p}_2+m\not{p}_2)P_L\Big] \text{ Tr}\Big[(-\not{p}_{1'}\not{p}_1+m\not{p}_1)P_R\Big] - 2m^2 \text{ Tr}\Big[(\not{p}_2)(\not{p}_1)P_R\Big]$$

$$+ \text{ Tr}\Big[(-\not{p}_{2'}\not{p}_2+m\not{p}_2)P_R\Big] \text{ Tr}\Big[(-\not{p}_{1'}\not{p}_1+m\not{p}_1)P_L\Big] - 2m^2 \text{ Tr}\Big[(\not{p}_2)(\not{p}_1)P_L\Big]$$

$$+ \text{ Tr}\Big[(-\not{p}_{1'}\not{p}_2+m\not{p}_2)P_L\Big] \text{ Tr}\Big[(-\not{p}_{2'}\not{p}_1+m\not{p}_1)P_R\Big]$$

$$+ \text{ Tr}\Big[(-\not{p}_{1'}\not{p}_2+m\not{p}_2)P_R\Big] \text{ Tr}\Big[(-\not{p}_{2'}\not{p}_1+m\not{p}_1)P_L\Big]\Big\}$$

Now recall that $P_{L,R} = \frac{1}{2}(1 \mp \gamma_5)$. Thus:

$$\langle|\mathcal{T}|^2\rangle = \frac{e^4}{4M^4}\Big\{ \text{ Tr}\Big[(-\not{p}_{2'}\not{p}_2+m\not{p}_2)(1-\gamma_5)\Big] \text{ Tr}\Big[(-\not{p}_{1'}\not{p}_1+m\not{p}_1)(1+\gamma_5)\Big] - 2m^2 \text{ Tr}\Big[(\not{p}_2)(\not{p}_1)(1+\gamma_5)\Big]$$

$$+ \text{ Tr}\Big[(-\not{p}_{2'}\not{p}_2+m\not{p}_2)(1+\gamma_5)\Big] \text{ Tr}\Big[(-\not{p}_{1'}\not{p}_1+m\not{p}_1)(1-\gamma_5)\Big] - 2m^2 \text{ Tr}\Big[(\not{p}_2)(\not{p}_1)(1-\gamma_5)\Big]$$

$$+ \text{ Tr}\Big[(-\not{p}_{1'}\not{p}_2+m\not{p}_2)(1-\gamma_5)\Big] \text{ Tr}\Big[(-\not{p}_{2'}\not{p}_1+m\not{p}_1)(1+\gamma_5)\Big]$$

$$+ \text{ Tr}\left[(-\not{p}_{1'}\not{p}_2 + m\not{p}_2)(1+\gamma_5)\right] \text{ Tr}\left[(-\not{p}_{2'}\not{p}_1 + m\not{p}_1)(1-\gamma_5)\right]\Big\}$$

Now many of these cancel. Recall that an odd number of gamma matrices cancel, an odd number of gamma matrices with a $\gamma_5$ cancels, and two gamma matrices with a $\gamma_5$ cancels. Thus:

$$\langle|\mathcal{T}|^2\rangle = \frac{e^4}{4M^4}\Big\{ \text{ Tr}\left[-\not{p}_{2'}\not{p}_2\right] \text{ Tr}\left[-\not{p}_{1'}\not{p}_1\right] - 2m^2 \text{ Tr}\left[\not{p}_2\not{p}_1\right] + \text{ Tr}\left[-\not{p}_{2'}\not{p}_2\right] \text{ Tr}\left[-\not{p}_{1'}\not{p}_1\right] - 2m^2 \text{ Tr}\left[\not{p}_2\not{p}_1\right]$$

$$+ \text{ Tr}\left[-\not{p}_{1'}\not{p}_2\right] \text{ Tr}\left[-\not{p}_{2'}\not{p}_1\right] + \text{ Tr}\left[-\not{p}_{1'}\not{p}_2\right] \text{ Tr}\left[-\not{p}_{2'}\not{p}_1\right]\Big\}$$

Combining a few of these:

$$\langle|\mathcal{T}|^2\rangle = \frac{e^4}{2M^4}\Big\{ \text{ Tr}\left[\not{p}_{2'}\not{p}_2\right] \text{ Tr}\left[\not{p}_{1'}\not{p}_1\right] - 4m^2 \text{ Tr}\left[\not{p}_2\not{p}_1\right] + \text{ Tr}\left[\not{p}_{1'}\not{p}_2\right] \text{ Tr}\left[\not{p}_{2'}\not{p}_1\right]\Big\}$$

Now we solve these traces:

$$\langle|\mathcal{T}|^2\rangle = \frac{4e^4}{M^4}\left\{2(p_{2'}\cdot p_2)(p_{1'}\cdot p_1) + 2m^2(p_1\cdot p_2) + 2(p_{1'}\cdot p_2)(p_{2'}\cdot p_1)\right\}$$

Now we have:

$$s = -(p_1+p_2)^2 \implies s = -2(p_1\cdot p_2) \implies p_1\cdot p_2 = \frac{-s}{2}$$

$$t = -(p_1-p_{1'})^2 \implies t = m^2 + 2(p_1\cdot p_{1'}) \implies p_1\cdot p_{1'} = \frac{t-m^2}{2} \tag{49.1.1}$$

$$t = -(p_2-p_{2'})^2 \implies t = m^2 + 2(p_2\cdot p_{2'}) \implies p_2\cdot p_{2'} = \frac{t-m^2}{2}$$

$$u = -(p_1-p_{2'})^2 \implies u = m^2 + 2(p_1\cdot p_{2'}) \implies p_1\cdot p_{2'} = \frac{u-m^2}{2} \tag{49.1.2}$$

$$u = -(p_{1'}-p_2)^2 \implies u = m^2 + 2(p_2\cdot p_{1'}) \implies p_2\cdot p_{1'} = \frac{u-m^2}{2}$$

Then:

$$\langle|\mathcal{T}|^2\rangle = \frac{2e^4}{M^4}\left\{(t-m^2)^2 - 2m^2 s + (u-m^2)^2\right\}$$

Now we want to calculate the cross-section, using 11.31:

$$\frac{d\sigma}{d\Omega_{CM}} = \frac{1}{64\pi^2 s}\frac{|k_{1'}|}{|k_1|}|\mathcal{T}|^2$$

We take the spin-average, and use 11.2 and 11.3:

$$\langle\frac{d\sigma}{d\Omega}\rangle_{CM} = \frac{1}{64\pi^2 s}\frac{\sqrt{s-4m^2}/2}{\sqrt{s}/2}\left[\frac{2e^4}{M^4}\left\{(t-m^2)^2 - 2m^2 s + (u-m^2)^2\right\}\right]$$

Simplifying:

$$\langle\frac{d\sigma}{d\Omega}\rangle_{CM} = \frac{e^4}{32\pi^2 M^4 s}\sqrt{\frac{s-4m^2}{s}}\left[(t-m^2)^2 - 2m^2 s + (u-m^2)^2\right]$$

Now we need to get rid of t, u and use the Polar Angle instead. We use (49.1.1):

$$t - m^2 = 2p_1 \cdot p_1' = -2E_1 E_{1'} + 2\vec{p}_1 \cdot \vec{p}_{1'}$$

Now we know the initial particles have no mass in our approximation, so $E_1 = |\vec{p}_1|$. Further, $E_{1'} = \sqrt{|\vec{p}_{1'}|^2 + m^2}$. Finally, we write out the dot product:

$$t - m^2 = 2p_1 \cdot p_1' = -2|\vec{p}_1|\sqrt{|\vec{p}_{1'}|^2 + m^2} + 2|\vec{p}_1||\vec{p}_{1'}|\cos\theta$$

Now we use 11.2 and 11.3 again:

$$t - m^2 = -2\frac{\sqrt{s}}{2}\sqrt{\frac{s-4m^2}{4} + m^2} + 2\frac{\sqrt{s}}{2}\frac{\sqrt{s-4m^2}}{2}\cos\theta$$

Simplifying:

$$t - m^2 = -\frac{s}{2} + \frac{\sqrt{s^2 - 4m^2 s}}{2}\cos\theta$$

Now for u; we'll use (49.1.2). Everything is the same as before, except $\vec{p}_1 \cdot \vec{p}_{2'} = |\vec{p}_1||\vec{p}_{2'}|\cos(\pi - \theta) = -|\vec{p}_1||\vec{p}_{2'}|\cos\theta$. Thus:

$$u - m^2 = -\frac{s}{2} - \frac{\sqrt{s^2 - 4m^2 s}}{2}\cos\theta$$

Putting this together, we have:

$$\langle\frac{d\sigma}{d\Omega}\rangle_{CM} = \frac{e^4}{32\pi^2 M^2 s}\sqrt{\frac{s-4m^2}{s}}\left[\left(-\frac{s}{2} + \frac{\sqrt{s^2-4m^2 s}}{2}\cos\theta\right)^2 + \left(-\frac{s}{2} - \frac{\sqrt{s^2-4m^2 s}}{2}\cos\theta\right)^2 - 2m^2 s\right]$$

Now it's just some algebra to simplify this, the result is:

$$\langle\frac{d\sigma}{d\Omega}\rangle_{CM} = \frac{e^4 s}{64\pi^2 M^2}\left(1 - \frac{4m^2}{s}\right)^{3/2}\left(1 + \cos^2\theta\right)$$

which is a relatively elegant result, considering what a mess the problem was!