Srednicki Chapter 48

QFT Problems & Solutions

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Srednicki 48.1. The tedium of these calculations is greatly alleviated by making use of a symbolic manipulation program like Mathematica or Maple. One approach is brute force: compute 4×4 matrices like $\not{p}$ in the CM frame, and take their products and traces. If you are familiar with a symbolic-manipulation program, write one that does this. See if you can verify equations 48.26-48.29.

The key point is to calculate $\not{p}_i$. Once we have that, it is a trivial matter to tell your favorite program to do the multiplication and take the trace; I won't even bother to do that here.

We have $\not{p}_i = p^\mu \gamma_\mu$. The γ matrices are given in the text, I'll write them out explicitely here:

$$\gamma_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The other thing we need is the ps. Let's consider the case of collisions, since we are trying to reproduce equations 48.26-29. Each initial particle has a four-momentum given by:

$$p_i = (E_i, 0, 0, p_i)$$

Each final particle has a momentum given by:

$$p_{i'} = (E_{i'}, p_{i'} \sin\theta, 0, p_{i'} \cos\theta)$$

The magnitude of the three-momentum p_i is given by equation 11.2 or 11.3. The magnitude of the energy E_i is constrained by the requirement that the magnitude of one of these vectors must be $-m^2$. Thus:

$$p_1 = \left[\frac{s + m_1^2 - m_2^2}{2\sqrt{s}}, 0, 0, \frac{\sqrt{s - 2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2}}{2s} \right]$$

$$p_2 = \left[\frac{s - m_1^2 + m_2^2}{2\sqrt{s}}, 0, 0, -\frac{\sqrt{s - 2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2}}{2s}\right]$$

$$p_3 = \left[\frac{s + m_3^2 - m_4^2}{2\sqrt{s}}, \sin\theta\frac{\sqrt{s - 2(m_3^2 + m_4^2) + (m_3^2 - m_4^2)^2}}{2s}, 0, \cos\theta\frac{\sqrt{s - 2(m_3^2 + m_4^2) + (m_3^2 - m_4^2)^2}}{2s}\right]$$

$$p_4 = \left[\frac{s - m_3^2 + m_4^2}{2\sqrt{s}}, -\sin\theta\frac{\sqrt{s - 2(m_3^2 + m_4^2) + (m_3^2 - m_4^2)^2}}{2s}, 0, -\cos\theta\frac{\sqrt{s - 2(m_3^2 + m_4^2) + (m_3^2 - m_4^2)^2}}{2s}\right]$$

Now we just have to do the dot product (on the computer). Notice that we're dotting a vector of numbers by a vector of matrices: we therefore have to multiply the vector of numbers by the identity. Then, we have:

$$\not{p}_i = -(p_{i0}I)\gamma^0 + (p_{i1}I)\gamma^1 + (p_{i2}I)\gamma^2 + (p_{i3}I)\gamma^3$$

which is the slash, as promised. From here it is trivial to have the computer do the multiplication and traces.

Srednicki 48.2. Compute $\langle|\mathcal{T}|^2\rangle$ for $e^+e^- \to \phi\phi$. You should find that your result is the same as that for $e^-\phi \to e^-\phi$, but with $s \leftrightarrow t$, and an extra factor of minus one-half. This relationship is known as *crossing symmetry*. There is an overall minus sign for each fermion that is moved from the initial to the final state.

Srednicki started this one with equation 45.23:

$$i\mathcal{T} = \frac{1}{i}(ig)^2\overline{v}_{s_2}(\vec{p}_2)\left[\frac{-\not{p}_1 + \not{k}_1' + m}{-t + m^2} + \frac{-\not{p}_1 + \not{k}_2' + m}{-u + m^2}\right]u_{s_1}(\vec{p}_1)$$

We can write this as:

$$\mathcal{T} = g^2\overline{v}_{s_2}(\vec{p}_2)\left[\frac{-\not{p}_1 + \not{k}_1' + m}{-t + m^2} + \frac{-\not{p}_1 + \not{k}_2' + m}{-u + m^2}\right]u_{s_1}(\vec{p}_1)$$

We can use the Dirac Equation to simplify this further: $-\not{p}u = mu$. Thus:

$$\mathcal{T} = g^2\overline{v}_{s_2}(\vec{p}_2)\left[\frac{\not{k}_1' + 2m}{-t + m^2} + \frac{\not{k}_2' + 2m}{-u + m^2}\right]u_{s_1}(\vec{p}_1)$$

Now we need the conjugate:

$$\mathcal{T}^* = g^2\overline{u}_{s_1}(\vec{p}_1)\overline{\left[\frac{\not{k}_1' + 2m}{-t + m^2} + \frac{\not{k}_2' + 2m}{-u + m^2}\right]}v_{s_2}(\vec{p}_2)$$

Everything in the large bracketed term is a constant that will be unaffected by the barring – except for the $\not{k} = -k^0\gamma^0 + k^i\gamma^i$ terms. These are also constants except for the gamma matrices; however, we know from equation 38.15 that the gamma matrices are also unaffected by the barring. Thus:

$$\mathcal{T}^* = g^2 \overline{u}_{s_1}(\vec{p}_1) \left[\frac{\not{k}_1' + 2m}{-t + m^2} + \frac{\not{k}_2' + 2m}{-u + m^2} \right] v_{s_2}(\vec{p}_2)$$

Now we are ready to evaluate $|\mathcal{T}|^2 = \mathcal{T}\mathcal{T}^*$:

$$|\mathcal{T}|^2 = g^4 \overline{v}_{s_2}(\vec{p}_2) \left[\frac{\not{k}_1' + 2m}{-t + m^2} + \frac{\not{k}_2' + 2m}{-u + m^2} \right] u_{s_1}(\vec{p}_1) \overline{u}_{s_1}(\vec{p}_1) \left[\frac{\not{k}_1' + 2m}{-t + m^2} + \frac{\not{k}_2' + 2m}{-u + m^2} \right] v_{s_2}(\vec{p}_2)$$

Now let's write the last term in index notation (implied sum over α and β, as usual):

$$|\mathcal{T}|^2 = g^4 \overline{v}_{s_2}(\vec{p}_2) \left[\frac{\not{k}_1' + 2m}{-t + m^2} + \frac{\not{k}_2' + 2m}{-u + m^2} \right] u_{s_1}(\vec{p}_1) \overline{u}_{s_1}(\vec{p}_1)_\alpha \left[\frac{\not{k}_1' + 2m}{-t + m^2} + \frac{\not{k}_2' + 2m}{-u + m^2} \right]_{\alpha\beta} v_{s_2}(\vec{p}_2)_\beta$$

Now an individual component of the u spinor on the right hand side is just a number; this will commute with everything. Let's write it at the beginning of the equation:

$$|\mathcal{T}|^2 = g^4 v_{s_2}(\vec{p}_2)_\beta \overline{v}_{s_2}(\vec{p}_2) \left[\frac{\not{k}_1' + 2m}{-t + m^2} + \frac{\not{k}_2' + 2m}{-u + m^2} \right] u_{s_1}(\vec{p}_1) \overline{u}_{s_1}(\vec{p}_1)_\alpha \left[\frac{\not{k}_1' + 2m}{-t + m^2} + \frac{\not{k}_2' + 2m}{-u + m^2} \right]_{\alpha\beta}$$

Now our implied sum over β leads to a trace (we also drop the index notation on α, no longer needed):

$$|\mathcal{T}|^2 = g^4 \ \mathrm{Tr} \left\{ v_{s_2}(\vec{p}_2) \overline{v}_{s_2}(\vec{p}_2) \left[\frac{\not{k}_1' + 2m}{-t + m^2} + \frac{\not{k}_2' + 2m}{-u + m^2} \right] u_{s_1}(\vec{p}_1) \overline{u}_{s_1}(\vec{p}_1) \left[\frac{\not{k}_1' + 2m}{-t + m^2} + \frac{\not{k}_2' + 2m}{-u + m^2} \right] \right\}$$

Now we need to sum over the final states (s') and averaged over the initial states (s). There are two initial states for two initial particles, so we have to sum and divide by four. Then:

$$\langle |\mathcal{T}|^2 \rangle = \frac{g^4}{4} \sum_{s_1, s_2} \ \mathrm{Tr} \left\{ v_{s_2}(\vec{p}_2) \overline{v}_{s_2}(\vec{p}_2) \left[\frac{\not{k}_1' + 2m}{-t + m^2} + \frac{\not{k}_2' + 2m}{-u + m^2} \right] u_{s_1}(\vec{p}_1) \overline{u}_{s_1}(\vec{p}_1) \left[\frac{\not{k}_1' + 2m}{-t + m^2} + \frac{\not{k}_2' + 2m}{-u + m^2} \right] \right\}$$

Now we can use 46.8 and 46.13 to evaluate these sums:

$$\langle |\mathcal{T}|^2 \rangle = \frac{g^4}{4} \ \mathrm{Tr} \left\{ (-\not{p}_2 - m) \left[\frac{\not{k}_1' + 2m}{-t + m^2} + \frac{\not{k}_2' + 2m}{-u + m^2} \right] (-\not{p}_1 + m) \left[\frac{\not{k}_1' + 2m}{-t + m^2} + \frac{\not{k}_2' + 2m}{-u + m^2} \right] \right\}$$

This is the correct answer, but recall that we have to simplify this into constants and Mandelstam variables. This is going to be tedious. First we break this into the 16 individual terms:

$$\langle |\mathcal{T}|^2 \rangle = \frac{g^4}{4} \ \mathrm{Tr} \left\{ (-\not{p}_2) \frac{\not{k}_1' + 2m}{-t + m^2} (-\not{p}_1) \frac{\not{k}_1' + 2m}{-t + m^2} + (-\not{p}_2) \frac{\not{k}_1' + 2m}{-t + m^2} (-\not{p}_1) \frac{\not{k}_2' + 2m}{-u + m^2} + \right.$$

$$(-\not{p}_2) \frac{\not{k}_1' + 2m}{-t + m^2} (m) \frac{\not{k}_1' + 2m}{-t + m^2} + (-\not{p}_2) \frac{\not{k}_1' + 2m}{-t + m^2} (m) \frac{\not{k}_2' + 2m}{-u + m^2} + (-\not{p}_2) \frac{\not{k}_2' + 2m}{-u + m^2} (-\not{p}_1) \frac{\not{k}_1' + 2m}{-t + m^2} +$$

$$(-\not{p}_2)\frac{\not{k}'_2+2m}{-u+m^2}(-\not{p}_1)\frac{\not{k}'_2+2m}{-u+m^2}+(-\not{p}_2)\frac{\not{k}'_2+2m}{-u+m^2}(m)\frac{-\not{k}'_1+2m}{-t+m^2}+(-\not{p}_2)\frac{\not{k}'_2+2m}{-u+m^2}(m)\frac{\not{k}'_2+2m}{-u+m^2}+$$

$$(-m)\frac{\not{k}'_1+2m}{-t+m^2}(-\not{p}_1)\frac{\not{k}'_1+2m}{-t+m^2}+(-m)\frac{\not{k}'_1+2m}{-t+m^2}(-\not{p}_1)\frac{\not{k}'_2+2m}{-u+m^2}+(-m)\frac{\not{k}'_1+2m}{-t+m^2}(m)\frac{\not{k}'_1+2m}{-t+m^2}$$

$$+(-m)\frac{\not{k}'_1+2m}{-t+m^2}(m)\frac{\not{k}'_2+2m}{-u+m^2}+(-m)\frac{\not{k}'_2+2m}{-u+m^2}(-\not{p}_1)\frac{\not{k}'_1+2m}{-t+m^2}+(-m)\frac{\not{k}'_2+2m}{-u+m^2}(-\not{p}_1)\frac{\not{k}'_2+2m}{-u+m^2}+$$

$$(-m)\frac{\not{k}'_2+2m}{-u+m^2}(m)\frac{-\not{k}'_1+2m}{-t+m^2}+(-m)\frac{\not{k}'_2+2m}{-u+m^2}(m)\frac{\not{k}'_2+2m}{-u+m^2}\Bigg\}$$

Let's distribute the trace and clean up a little bit:

$$\langle|\mathcal{T}|^2\rangle=\frac{g^4}{4}\Bigg\{\ \mathrm{Tr}\left[(\not{p}_2)\frac{\not{k}'_1+2m}{-t+m^2}(\not{p}_1)\frac{\not{k}'_1+2m}{-t+m^2}\right]+\ \mathrm{Tr}\left[(\not{p}_2)\frac{\not{k}'_1+2m}{-t+m^2}(\not{p}_1)\frac{\not{k}'_2+2m}{-u+m^2}\right]-$$

$$\mathrm{Tr}\left[(\not{p}_2)\frac{\not{k}'_1+2m}{-t+m^2}(m)\frac{\not{k}'_1+2m}{-t+m^2}\right]-\ \mathrm{Tr}\left[(\not{p}_2)\frac{\not{k}'_1+2m}{-t+m^2}(m)\frac{\not{k}'_2+2m}{-u+m^2}\right]+\ \mathrm{Tr}\left[(\not{p}_2)\frac{\not{k}'_2+2m}{-u+m^2}(\not{p}_1)\frac{\not{k}'_1+2m}{-t+m^2}\right]+$$

$$\mathrm{Tr}\left[(\not{p}_2)\frac{\not{k}'_2+2m}{-u+m^2}(\not{p}_1)\frac{\not{k}'_2+2m}{-u+m^2}\right]-\ \mathrm{Tr}\left[(\not{p}_2)\frac{\not{k}'_2+2m}{-u+m^2}(m)\frac{\not{k}'_1+2m}{-t+m^2}\right]-\ \mathrm{Tr}\left[(\not{p}_2)\frac{\not{k}'_2+2m}{-u+m^2}(m)\frac{\not{k}'_2+2m}{-u+m^2}\right]+$$

$$\mathrm{Tr}\left[(m)\frac{\not{k}'_1+2m}{-t+m^2}(\not{p}_1)\frac{\not{k}'_1+2m}{-t+m^2}\right]+\ \mathrm{Tr}\left[(m)\frac{\not{k}'_1+2m}{-t+m^2}(\not{p}_1)\frac{\not{k}'_2+2m}{-u+m^2}\right]-\ \mathrm{Tr}\left[(m)\frac{\not{k}'_1+2m}{-t+m^2}(m)\frac{\not{k}'_1+2m}{-t+m^2}\right]-$$

$$\mathrm{Tr}\left[(m)\frac{\not{k}'_1+2m}{-t+m^2}(m)\frac{\not{k}'_2+2m}{-u+m^2}\right]+\ \mathrm{Tr}\left[(m)\frac{\not{k}'_2+2m}{-u+m^2}(\not{p}_1)\frac{\not{k}'_1+2m}{-t+m^2}\right]+\ \mathrm{Tr}\left[(m)\frac{\not{k}'_2+2m}{-u+m^2}(\not{p}_1)\frac{\not{k}'_2+2m}{-u+m^2}\right]$$

$$-\ \mathrm{Tr}\left[(m)\frac{\not{k}'_2+2m}{-u+m^2}(m)\frac{\not{k}'_1+2m}{-t+m^2}\right]-\ \mathrm{Tr}\left[(m)\frac{\not{k}'_2+2m}{-u+m^2}(m)\frac{\not{k}'_2+2m}{-u+m^2}\right]\Bigg\}$$

Let's pull out as many scalars as possible:

$$\langle|\mathcal{T}|^2\rangle=\frac{g^4}{4}\Bigg\{\frac{\mathrm{Tr}\left[(\not{p}_2)(\not{k}'_1+2m)(\not{p}_1)(\not{k}'_1+2m)\right]}{(-t+m^2)(-t+m^2)}+\frac{\mathrm{Tr}\left[(\not{p}_2)(\not{k}'_1+2m)(\not{p}_1)(\not{k}'_2+2m)\right]}{(-t+m^2)(-u+m^2)}-$$

$$m\frac{\mathrm{Tr}\left[(\not{p}_2)(\not{k}'_1+2m)(\not{k}'_1+2m)\right]}{(-t+m^2)(-t+m^2)}-m\frac{\mathrm{Tr}\left[(\not{p}_2)(\not{k}'_1+2m)(\not{k}'_2+2m)\right]}{(-t+m^2)(-u+m^2)}+\frac{\mathrm{Tr}\left[(\not{p}_2)(\not{k}'_2+2m)(\not{p}_1)(\not{k}'_1+2m)\right]}{(-u+m^2)(-t+m^2)}+$$

$$\frac{\mathrm{Tr}\left[(\not{p}_2)(\not{k}'_2+2m)(\not{p}_1)(\not{k}'_2+2m)\right]}{(-u+m^2)(-u+m^2)}-m\frac{\mathrm{Tr}\left[(\not{p}_2)(\not{k}'_2+2m)(\not{k}'_1+2m)\right]}{(-u+m^2)(-t+m^2)}-m\frac{\mathrm{Tr}\left[(\not{p}_2)(\not{k}'_2+2m)(\not{k}'_2+2m)\right]}{(-u+m^2)(-u+m^2)}+$$

$$m\frac{\mathrm{Tr}\left[(\not{k}'_1+2m)(\not{p}_1)(\not{k}'_1+2m)\right]}{(-t+m^2)(-t+m^2)}+m\frac{\mathrm{Tr}\left[(\not{k}'_1+2m)(\not{p}_1)(\not{k}'_2+2m)\right]}{(-t+m^2)(-u+m^2)}-m^2\frac{\mathrm{Tr}\left[(\not{k}'_1+2m)(\not{k}'_1+2m)\right]}{(-t+m^2)(-t+m^2)}-$$

$$m^2\frac{\mathrm{Tr}\left[(\not{k}'_1+2m)(\not{k}'_2+2m)\right]}{(-t+m^2)(-u+m^2)}+m\frac{\mathrm{Tr}\left[(\not{k}'_2+2m)(\not{p}_1)(\not{k}'_1+2m)\right]}{(-u+m^2)(-t+m^2)}+m\frac{\mathrm{Tr}\left[(\not{k}'_2+2m)(\not{p}_1)(\not{k}'_2+2m)\right]}{(-u+m^2)(-u+m^2)}-$$

$$-m^2\frac{\mathrm{Tr}\left[(\not{k}_2'+2m)(\not{k}_1'+2m)\right]}{(-u+m^2)(-t+m^2)}-m^2\frac{\mathrm{Tr}\left[(\not{k}_2'+2m)(\not{k}_2'+2m)\right]}{(-u+m^2)(-u+m^2)}\Bigg\}$$

This is too big to handle; as in the book, we'll break this into four components:

$$\langle|\mathcal{T}|^2\rangle=\frac{g^4}{4}\left[\frac{\langle\Phi_{tt}\rangle}{(-t+m^2)(-t+m^2)}+\frac{\langle\Phi_{tu}\rangle}{(-t+m^2)(-u+m^2)}+\frac{\langle\Phi_{ut}\rangle}{(-u+m^2)(-t+m^2)}\right.$$
$$\left.+\frac{\langle\Phi_{uu}\rangle}{(-u+m^2)(-u+m^2)}\right]$$

Now it's just a matter of evaluating these. We have:

$$\langle\Phi_{tt}\rangle=\mathrm{Tr}\left[(\not{p}_2)(\not{k}_1'+2m)(\not{p}_1)(\not{k}_1'+2m)\right]-m\mathrm{Tr}\left[(\not{p}_2)(\not{k}_1'+2m)(\not{k}_1'+2m)\right]$$
$$+m\mathrm{Tr}\left[(\not{k}_1'+2m)(\not{p}_1)(\not{k}_1'+2m)\right]-m^2\mathrm{Tr}\left[(\not{k}_1'+2m)(\not{k}_1'+2m)\right]$$
$$\langle\Phi_{tu}\rangle=\mathrm{Tr}\left[(\not{p}_2)(\not{k}_1'+2m)(\not{p}_1)(\not{k}_2'+2m)\right]-m\mathrm{Tr}\left[(\not{p}_2)(\not{k}_1'+2m)(\not{k}_2'+2m)\right]$$
$$+m\mathrm{Tr}\left[(\not{k}_1'+2m)(\not{p}_1)(\not{k}_2'+2m)\right]-m^2\mathrm{Tr}\left[(\not{k}_1'+2m)(\not{k}_2'+2m)\right]$$
$$\langle\Phi_{ut}\rangle=\mathrm{Tr}\left[(\not{p}_2)(\not{k}_2'+2m)(\not{p}_1)(\not{k}_1'+2m)\right]-m\mathrm{Tr}\left[(\not{p}_2)(\not{k}_2'+2m)(\not{k}_1'+2m)\right]$$
$$+m\mathrm{Tr}\left[(\not{k}_2'+2m)(\not{p}_1)(\not{k}_1'+2m)\right]-m^2\mathrm{Tr}\left[(\not{k}_2'+2m)(\not{k}_1'+2m)\right]$$
$$\langle\Phi_{uu}\rangle=\mathrm{Tr}\left[(\not{p}_2)(\not{k}_2'+2m)(\not{p}_1)(\not{k}_2'+2m)\right]-m\mathrm{Tr}\left[(\not{p}_2)(\not{k}_2'+2m)(\not{k}_2'+2m)\right]$$
$$+m\mathrm{Tr}\left[(\not{k}_2'+2m)(\not{p}_1)(\not{k}_2'+2m)\right]-m^2\mathrm{Tr}\left[(\not{k}_2'+2m)(\not{k}_2'+2m)\right]$$

$\Phi_{tt}\to\Phi_{uu}$ with $k_1'\to k_2'$. Further, $\Phi_{tu}\to\Phi_{ut}$ with $k_1'\leftrightarrow k_2'$. Therefore, we need to keep only the first two of these, then we can write down the result for the other two.

Next we'll break up these binomials; this will turn each of these from 4 terms to 16. However, half of them will have an odd number of gamma matrices, the trace of which vanishes. Thus, this reduces to:

$$\langle\Phi_{tt}\rangle=\mathrm{Tr}\left[(\not{p}_2)(\not{k}_1')(\not{p}_1)(\not{k}_1')\right]-2m^2\mathrm{Tr}\left[(\not{p}_2)(\not{k}_1')\right]+2m^2\mathrm{Tr}\left[(\not{k}_1')(\not{p}_1)\right]-m^2\mathrm{Tr}\left[(\not{k}_1')(\not{k}_1')\right]$$
$$+4m^2\mathrm{Tr}\left[(\not{p}_2)(\not{p}_1)\right]-2m^2\mathrm{Tr}\left[(\not{p}_2)(\not{k}_1')\right]+2m^2\mathrm{Tr}\left[(\not{p}_1)(\not{k}_1')\right]-4m^4\mathrm{Tr}\left[1\right]$$
$$\langle\Phi_{tu}\rangle=\mathrm{Tr}\left[(\not{p}_2)(\not{k}_1')(\not{p}_1)(\not{k}_2')\right]-2m^2\mathrm{Tr}\left[(\not{p}_2)(\not{k}_1')\right]+2m^2\mathrm{Tr}\left[(\not{k}_1')(\not{p}_1)\right]-m^2\mathrm{Tr}\left[(\not{k}_1')(\not{k}_2')\right]$$
$$+4m^2\mathrm{Tr}\left[(\not{p}_2)(\not{p}_1)\right]-2m^2\mathrm{Tr}\left[(\not{p}_2)(\not{k}_2')\right]+2m^2\mathrm{Tr}\left[(\not{p}_1)(\not{k}_2')\right]-4m^4\mathrm{Tr}\left[1\right]$$

We can simplify this a bit:

$$\langle\Phi_{tt}\rangle=\mathrm{Tr}\left[(\not{p}_2)(\not{k}_1')(\not{p}_1)(\not{k}_1')\right]-4m^2\mathrm{Tr}\left[(\not{p}_2)(\not{k}_1')\right]+4m^2\mathrm{Tr}\left[(\not{k}_1')(\not{p}_1)\right]-m^2\mathrm{Tr}\left[(\not{k}_1')(\not{k}_1')\right]$$

$$+4m^2\mathrm{Tr}\left[(\not{p}_2)(\not{p}_1)\right]-16m^4$$

$$\langle\Phi_{tu}\rangle=\mathrm{Tr}\left[(\not{p}_2)(\not{k}'_1)(\not{p}_1)(\not{k}'_2)\right]-2m^2\mathrm{Tr}\left[(\not{p}_2)(\not{k}'_1)\right]+2m^2\mathrm{Tr}\left[(\not{k}'_1)(\not{p}_1)\right]-m^2\mathrm{Tr}\left[(\not{k}'_1)(\not{k}'_2)\right]$$

$$+4m^2\mathrm{Tr}\left[(\not{p}_2)(\not{p}_1)\right]-2m^2\mathrm{Tr}\left[(\not{p}_2)(\not{k}'_2)\right]+2m^2\mathrm{Tr}\left[(\not{p}_1)(\not{k}'_2)\right]-16m^4$$

Now we are ready to use equations 47.9 and 47.13:

$$\langle\Phi_{tt}\rangle=4(p_2\cdot k'_1)(k'_1\cdot p_1)-4(p_2\cdot p_1)(k'_1\cdot k'_1)+4(p_2\cdot k'_1)(p_1\cdot k'_1)+16m^2(p_2\cdot k'_1)-16m^2(k'_1\cdot p_1)+4m^2(k'_1\cdot k'_1)$$

$$-16m^2(p_2\cdot p_1)-16m^4$$

$$\langle\Phi_{tu}\rangle=4(p_2\cdot k'_2)(k'_1\cdot p_1)-4(p_2\cdot p_1)(k'_1\cdot k'_2)+4(p_2\cdot k'_1)(p_1\cdot k'_2)+8m^2(p_2\cdot k'_1)-8m^2(k'_1\cdot p_1)+4m^2(k'_1\cdot k'_2)$$

$$-16m^2(p_2\cdot p_1)+8m^2(p_2\cdot k'_2)-8m^2(p_1\cdot k'_2)-16m^4$$

We also have $k'\cdot k'=-M^2$:

$$\langle\Phi_{tt}\rangle=8(p_2\cdot k'_1)(k'_1\cdot p_1)+4M^2(p_2\cdot p_1)+16m^2(p_2\cdot k'_1)-16m^2(k'_1\cdot p_1)-4m^2M^2$$

$$-16m^2(p_2\cdot p_1)-16m^4$$

$$\langle\Phi_{tu}\rangle=4(p_2\cdot k'_2)(k'_1\cdot p_1)-4(p_2\cdot p_1)(k'_1\cdot k'_2)+4(p_2\cdot k'_1)(p_1\cdot k'_2)+8m^2(p_2\cdot k'_1)-8m^2(k'_1\cdot p_1)+4m^2(k'_1\cdot k'_2)$$

$$-16m^2(p_2\cdot p_1)+8m^2(p_2\cdot k'_2)-8m^2(p_1\cdot k'_2)-16m^4$$

Now we can start to use Mandelstam Variables:

$$s=-(p_1+p_2)^2=2m^2-2(p_1\cdot p_2)$$

$$s=-(k'_1+k'_2)^2=2M^2-2(k'_1\cdot k'_2)$$

$$t=-(p_1-k'_1)^2=m^2+M^2+2(p_1\cdot k'_1)$$

$$t=-(p_2-k'_2)^2=m^2+M^2+2(p_2\cdot k'_2)$$

$$u=-(p_1-k'_2)^2=m^2+M^2+2(p_1\cdot k'_2)$$

$$u=-(p_2-k'_1)^2=m^2+M^2+2(p_2\cdot k'_1)$$

These give:

$$2(p_1\cdot p_2)=2m^2-s$$

$$2(k'_1\cdot k'_2)=2M^2-s$$

$$2(p_1\cdot k'_1)=t-m^2-M^2$$

$$2(p_2\cdot k'_2)=t-m^2-M^2$$

$$2(p_1\cdot k'_2)=u-m^2-M^2$$

$$2(p_2\cdot k'_1)=u-m^2-M^2$$

We insert these into our expressions for the Φs:

$$\langle\Phi_{tt}\rangle=2(u-m^2-M^2)(t-m^2-M^2)+2M^2(2m^2-s)+8m^2(u-m^2-M^2)-8m^2(t-m^2-M^2)-4m^2M^2$$

$$-8m^2(2m^2-s)-16m^4$$

$$\langle\Phi_{tu}\rangle = (t-m^2-M^2)(t-m^2-M^2)-(2m^2-s)(2M^2-s)+(u-m^2-M^2)(u-m^2-M^2)+4m^2(u-m^2-M^2)$$
$$-4m^2(t-m^2-M^2)+2m^2(2M^2-s)-8m^2(2m^2-s)+4m^2(t-m^2-M^2)-4m^2(u-M^2-m^2)-16m^4$$

Now we just have to simplify this, the result of the algebra is:

$$\langle\Phi_{tt}\rangle = ut - m^2(-3u+5t-4s) - 15m^2 - M^2(s+t+u) + 2m^2M^2 + M^4$$

$$\langle\Phi_{tu}\rangle = t^2+u^2-s^2+2m^2(4s-t-u)+2M^2(s-t-u)-30m^4+4m^2M^2+2M^4$$

Now we use $s+t+u = 2m^2+2M^2$ and simplify further, the result is:

$$\langle\Phi_{tt}\rangle = 2\left[tu - m^2(9t+u) - 7m^4 + 8m^2M^2 - M^4\right]$$

$$\langle\Phi_{tu}\rangle = 2\left[tu + 3m^2(t+u) + 9m^4 - 8m^2M^2 - M^4\right]$$

We get the remaining terms by swapping $k_1' \leftrightarrow k_2'$, which we see from above is $t \leftrightarrow u$. Then:

$$\langle\Phi_{uu}\rangle = 2\left[ut - m^2(9u+t) - 7m^4 + 8m^2M^2 - M^4\right]$$

$$\langle\Phi_{ut}\rangle = 2\left[ut + 3m^2(t+u) + 9m^4 - 8m^2M^2 - M^4\right]$$

Putting all this together, the scattering amplitude is:

$$\langle|\mathcal{T}|^2\rangle = \frac{g^4}{2}\left[\frac{tu - m^2(9t+u) - 7m^4 + 8m^2M^2 - M^4}{(-t+m^2)(-t+m^2)} + \frac{2tu + 6m^2(t+u) + 18m^4 - 16m^2M^2 - 2M^4}{(-t+m^2)(-u+m^2)}\right.$$
$$\left.+\frac{ut - m^2(9u+t) - 7m^4 + 8m^2M^2 - M^4}{(-u+m^2)(-u+m^2)}\right]$$

which is not a very intuitive result, but there it is.

Srednicki 48.3. Compute $\langle|\mathcal{T}|^2\rangle$ for $e^-e^- \to e^-e^-$. You should find that your result is the same as that for $e^+e^- \to e^+e^-$ but with $s \leftrightarrow u$. This is another example of crossing symmetry.

Srednicki started this one with 45.24:

$$i\mathcal{T} = \frac{1}{i}(ig)^2\left[\frac{(\overline{u}_1'u_1)(\overline{u}_2'u_2)}{-t+M^2} - \frac{(\overline{u}_2'u_1)(\overline{u}_1'u_2)}{-u+M^2}\right]$$

This gives:

$$\mathcal{T} = g^2\left[\frac{(\overline{u}_1'u_1)(\overline{u}_2'u_2)}{-t+M^2} - \frac{(\overline{u}_2'u_1)(\overline{u}_1'u_2)}{-u+M^2}\right]$$

Taking the conjugate of this:

$$\mathcal{T}^* = g^2\left[\frac{(\overline{u}_1u_1')(\overline{u}_2u_2')}{-t+M^2} - \frac{(\overline{u}_1u_2')(\overline{u}_2u_1')}{-u+M^2}\right]$$

This gives:

$$|\mathcal{T}|^2 = g^4 \left[\frac{(\overline{u}_1' u_1)(\overline{u}_2' u_2)}{-t+M^2} - \frac{(\overline{u}_2' u_1)(\overline{u}_1' u_2)}{-u+M^2} \right] \left[\frac{(\overline{u}_1 u_1')(\overline{u}_2 u_2')}{-t+M^2} - \frac{(\overline{u}_1 u_2')(\overline{u}_2 u_1')}{-u+M^2} \right]$$

Distributing:

$$|\mathcal{T}|^2 = g^4 \left[\frac{\Phi_{tt}}{(-t+M^2)(-t+M^2)} - \frac{\Phi_{tu}}{(-t+M^2)(-u+M^2)} - \frac{\Phi_{ut}}{(-u+M^2)(-u+M^2)} + \frac{\Phi_{uu}}{(-u+M^2)(-u+M^2)} \right]$$

where:

$$\Phi_{tt} = (\overline{u}_1' u_1)(\overline{u}_2' u_2)(\overline{u}_1 u_1')(\overline{u}_2 u_2')$$
$$\Phi_{tu} = (\overline{u}_1' u_1)(\overline{u}_2' u_2)(\overline{u}_1 u_2')(\overline{u}_2 u_1')$$
$$\Phi_{ut} = (\overline{u}_2' u_1)(\overline{u}_1' u_2)(\overline{u}_1 u_1')(\overline{u}_2 u_2')$$
$$\Phi_{ut} = (\overline{u}_2' u_1)(\overline{u}_1' u_2)(\overline{u}_1 u_2')(\overline{u}_2 u_1')$$

Note that swapping $u_1' \leftrightarrow u_2'$ and $\overline{u}_1' \leftrightarrow \overline{u}_2'$ will exchange $\Phi_{tt} \leftrightarrow \Phi_{uu}$ and $\Phi_{tu} \leftrightarrow \Phi_{ut}$. It is therefore only necessary to evaluate two of these.

Now we use our usual trick of writing this as a trace. First though, let's put this in the order that will allow us to use the completeness relations:

$$\Phi_{tt} = (\overline{u}_1' u_1)(\overline{u}_1 u_1')(\overline{u}_2' u_2)(\overline{u}_2 u_2')$$

$$\Phi_{tu} = (\overline{u}_1' u_1)(\overline{u}_1 u_2')(\overline{u}_2' u_2)(\overline{u}_2 u_1')$$

Now we are ready to use our usual trick of writing the last term in index notation, moving it to the front, recognizing this as the trace, then using the cyclic property of the trace to move this term back to its original position. In the case of Φ_{tt}, we need two seperate traces, as there is no way to use the completeness relations if we only take one trace. Thus:

$$\Phi_{tt} = \ \mathrm{Tr}\left[(\overline{u}_1' u_1)(\overline{u}_1 u_1')\right] \ \mathrm{Tr}\left[(\overline{u}_2' u_2)(\overline{u}_2 u_2')\right]$$

$$\Phi_{tu} = \ \mathrm{Tr}\left[(\overline{u}_1' u_1)(\overline{u}_1 u_2')(\overline{u}_2' u_2)(\overline{u}_2 u_1')\right]$$

Using the cyclic property gives:

$$\Phi_{tt} = \ \mathrm{Tr}\left[u_1' \overline{u}_1' u_1 \overline{u}_1\right] \ \mathrm{Tr}\left[u_2' \overline{u}_2' u_2 \overline{u}_2\right]$$

$$\Phi_{tu} = \ \mathrm{Tr}\left[u_1' \overline{u}_1' u_1 \overline{u}_1 u_2' \overline{u}_2' u_2 \overline{u}_2\right]$$

Now we will average over the four initial states, and sum over the final states:

$$\langle \Phi_{tt} \rangle = \frac{1}{4} \sum_{1,2,1',2'} \ \mathrm{Tr}\left[u_1' \overline{u}_1' u_1 \overline{u}_1\right] \ \mathrm{Tr}\left[u_2' \overline{u}_2' u_2 \overline{u}_2\right]$$

$$\langle\Phi_{tu}\rangle = \frac{1}{4}\sum_{1,2,1',2'} \text{Tr}\left[u_1'\overline{u}_1'u_1\overline{u}_1u_2'\overline{u}_2'u_2\overline{u}_2\right]$$

Now we can use the completeness relations:

$$\langle\Phi_{tt}\rangle = \frac{1}{4}\ \text{Tr}\left[(-\not{p}_1' + m)(-\not{p}_1 + m)\right]\ \text{Tr}\left[(-\not{p}_2' + m)(-\not{p}_2 + m)\right]$$

$$\langle\Phi_{tu}\rangle = \frac{1}{4}\ \text{Tr}\left[(-\not{p}_1' + m)(-\not{p}_1 + m)(-\not{p}_2' + m)(-\not{p}_2 + m)\right]$$

Now we have to distribute these, but recall that the trace of an odd number of gamma matrices vanishes. As a result, we can neglect half of our terms:

$$\langle\Phi_{tt}\rangle = \left[\frac{1}{4}\ \text{Tr}[\not{p}_1'\not{p}_1] + \frac{m^2}{4}\ \text{Tr}[1]\right]\left[\ \text{Tr}\left[\not{p}_2'\not{p}_2\right] + m^2\ \text{Tr}[1]\right]$$

$$\langle\Phi_{tu}\rangle = \frac{1}{4}\ \text{Tr}\left[\not{p}_1'\not{p}_1\not{p}_2'\not{p}_2\right] + \frac{m^2}{4}\left[\not{p}_1'\not{p}_1 + \not{p}_1'\not{p}_2' + \not{p}_1'\not{p}_2 + \not{p}_1\not{p}_2' + \not{p}_1\not{p}_2 + \not{p}_2'\not{p}_2\right] + \frac{1}{4}m^4\ \text{Tr}[1]$$

Simplifying this, we have:

$$\langle\Phi_{tt}\rangle = \frac{1}{4}\ \text{Tr}[\not{p}_1'\not{p}_1]\ \text{Tr}[\not{p}_2'\not{p}_2] + \frac{m^2}{4}\ \text{Tr}[1]\ \text{Tr}[\not{p}_1'\not{p}_1 + \not{p}_2'\not{p}_2] + \frac{m^4}{4}\ \text{Tr}[1]\ \text{Tr}[1]$$

$$\langle\Phi_{tu}\rangle = \frac{1}{4}\ \text{Tr}\left[\not{p}_1'\not{p}_1\not{p}_2'\not{p}_2\right] + \frac{m^2}{4}\left[\not{p}_1'\not{p}_1 + \not{p}_1'\not{p}_2' + \not{p}_1'\not{p}_2 + \not{p}_1\not{p}_2' + \not{p}_1\not{p}_2 + \not{p}_2'\not{p}_2\right] + \frac{1}{4}m^4\ \text{Tr}[1]$$

This gives:

$$\langle\Phi_{tt}\rangle = 4(p_1\cdot p_1')(p_2\cdot p_2') - 4m^2(p_1\cdot p_1') - 4m^2(p_2\cdot p_2') + 4m^4$$

$$\langle\Phi_{tu}\rangle = (p_1'\cdot p_2)(p_1\cdot p_2') - (p_1'\cdot p_2')(p_1\cdot p_2) + (p_1'\cdot p_1)(p_2'\cdot p_2) - m^2(p_1'\cdot p_1 + p_1'\cdot p_2' + p_1'\cdot p_2 + p_1\cdot p_2' + p_1\cdot p_2 + p_2'\cdot p_2) + m^4$$

This time the Mandelstam variables are given by equation 48.11. Plugging these into our expressions for the Φs, we have:

$$\langle\Phi_{tt}\rangle = (t - 2m^2)^2 - 4m^2(t - 2m^2) + 4m^4$$

$$\langle\Phi_{tu}\rangle = \frac{1}{4}(u-2m^2)^2 - \frac{1}{4}(s-2m^2)^2 + \frac{1}{4}(t-2m^2)^2 - \frac{1}{2}m^2\left[(t - 2m^2) + (2m^2 - s) + (u - 2m^2) + \right.$$
$$\left.(u - 2m^2) + (2m^2 - s) + (t - 2m^2)\right] + m^4$$

Simplifying, and making use of $s + t + u = 4m^2$:

$$\langle\Phi_{tt}\rangle = (t - 4m^2)^2$$

$$\langle\Phi_{tu}\rangle = \frac{1}{2}(-tu + 4m^2 s)$$

Now to get the remaining two Φs. We have to exchange $u_1' \leftrightarrow u_2'$ and $\overline{u}_1' \leftrightarrow \overline{u}_2'$, which by the completeness relations means swapping $p_1' \leftrightarrow p_2'$. According to 48.11, this means swapping $t \leftrightarrow u$. Thus:

$$\langle\Phi_{uu}\rangle = (u - 4m^2)^2$$

$$\langle \Phi_{ut} \rangle = \frac{1}{2}(-tu + 4m^2 s)$$

Combining all our results, we have:

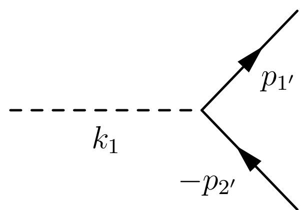
$$\langle |\mathcal{T}|^2 \rangle = g^4 \left[\frac{(t-4m^2)^2}{(-t+M^2)^2} + \frac{tu - 4m^2 s}{(-t+M^2)(-u+M^2)} + \frac{(u-4m^2)^2}{(-u+M^2)^2} \right]$$

Again, not exactly a beautiful result, but there we are.

Srednicki 48.4. Suppose that M $\geq$ 2m, so that the scalar can decay to an electron-positron pair.

(a) Compute the decay rate, summed over final spins.

Let's draw the diagram:



We use the Feynman rules to assess the value of this diagram:

$$i\mathcal{T} = \overline{u}_{s_{1'}}(\vec{p}_{1'})(ig)(1)v_{s_{2'}}(\vec{p}_{2'})$$

The magnitude of this is:

$$|\mathcal{T}|^2 = g^2 \left[\overline{u}_{s_1'}(\vec{p}_{1'}) v_{s_{2'}}(\vec{p}_{2'}) \overline{v}_{s_{2'}}(\vec{p}_{2'}) u_{s_{1'}}(\vec{p}_{1'}) \right]$$

Now we use our usual trace trick, giving us:

$$|\mathcal{T}|^2 = g^2 \text{ Tr} \left[u_{s_{1'}}(\vec{p}_{1'}) \overline{u}_{s_1'}(\vec{p}_{1'}) v_{s_{2'}}(\vec{p}_{2'}) \overline{v}_{s_{2'}}(\vec{p}_{2'}) \right] \tag{48.4.1}$$

Using the completeness relation:

$$\langle |\mathcal{T}|^2 \rangle = g^2 \text{ Tr} \left[(-\not{p}_{1'} + m)(-\not{p}_{2'} - m) \right]$$

We can simplify this:

$$\langle |\mathcal{T}|^2 \rangle = g^2 \text{ Tr} \left(\not{p}_{1'} \not{p}_{2'} \right) - g^2 m^2 Tr(1)$$

Using our trace identities:

$$\langle |\mathcal{T}|^2 \rangle = -4g^2 (p_{1'} \cdot p_{2'}) - 4m^2 g^2$$

Recall the definition of the Mandelstam variables:

$$s = -(p_{1'} + p_{2'})^2 = 2m^2 - 2(p_{1'} \cdot p_{2'}) \implies p_{1'} \cdot p_{2'} = m^2 - \frac{s}{2} \tag{48.4.2}$$

Using this, we have:

$$\langle|\mathcal{T}|^2\rangle = 2g^2(s - 4m^2)$$

For decay processes, recall that $s = M^2$. Thus:

$$\langle|\mathcal{T}|^2\rangle = 2g^2(M^2 - 4m^2) \tag{48.4.3}$$

Now we can use this in equation 11.48:

$$d\Gamma = \frac{1}{2M}2g^2(M^2 - 4m^2)\frac{|k_{1'}|}{16\pi^2\sqrt{s}}d\Omega_{CM}$$

Now we use equation 11.3:

$$|k_{1'}| = \frac{1}{2M}\sqrt{M^4 - 4m^2M^2} = \frac{1}{2}\sqrt{M^2 - 4m^2}$$

Putting this all together:

$$d\Gamma = \frac{1}{2M}2g^2(M^2 - 4m^2)\frac{1}{2}\sqrt{M^2 - 4m^2}\frac{1}{16\pi^2 M}d\Omega_{CM}$$

Simplifying:

$$d\Gamma = \frac{g^2}{32\pi^2M^2}(M^2 - 4m^2)^{3/2}d\Omega$$

The symmetry factor for the diagram is 1, so we use 11.49:

$$\Gamma = \frac{g^2}{8\pi M^2}(M^2 - 4m^2)^{3/2}$$

which is:

$$\Gamma = \frac{g^2M}{8\pi}\left[1 - \left(\frac{2m}{M}\right)^2\right]^{3/2}$$

(b) Compute $|\mathcal{T}|^2$ for decay into an electron with spin s_1 and a positron with spin s_2. Take the fermion three-momenta to be along the z-axis, and let the x-axis be the spin-quantization axis. You should find that $|\mathcal{T}|^2 = 0$ if the $s_1 = -s_2$. Discuss this in light of conservation of angular momentum and of parity.

This is the same problem as the previous one, except this time we will not sum or average over the spin states. Thus, we follow the derivation from part (a) up until the summing and averaging (equation (48.4.1)):

$$|\mathcal{T}|^2 = g^2 \text{ Tr}\left[u_{s_{1'}}(\vec{p}_{1'})\overline{u}_{s'_1}(\vec{p}_{1'})v_{s_{2'}}(\vec{p}_{2'})\overline{v}_{s_{2'}}(\vec{p}_{2'})\right]$$

Now we can use equation 38.28, but x is the spin-quantization axis, so we must take $z \to x$. Thus:

$$|\mathcal{T}|^2 = \frac{g^2}{4} \text{ Tr}\left[(1 - s_1\gamma_5\not{x})(-p_{1'} + m)(1 - s_2\gamma_5\not{x})(-\not{p}_{2'} - m)\right] \tag{48.4.4}$$

Now we simplify, dropping those terms with an odd number of γ matrices:

$$|\mathcal{T}|^2 = \frac{g^2}{4}\ \text{Tr}\left[(-\not{p}_{1'})(-\not{p}_{2'}) + (-\not{p}_{1'})(-s_{2'}\gamma_5\not{x})(-m) - m^2 + m(-s_{2'}\gamma_5\not{x})(-\not{p}_{2'})\right.$$
$$+(-s_{1'}\gamma_5\not{x})(-\not{p}_{1'})(-m) + (-s_{1'}\gamma_5\not{x})(-\not{p}_{1'})(-s_{2'}\gamma_5\not{x})(-\not{p}_{2'}) + (-s_{1'}\gamma_5\not{x})(m)(-\not{p}_{2'})$$
$$\left.+(-s_{1'}\gamma_5\not{x})(m)(-s_{2'}\gamma_5\not{x})(-m)\right]$$

We can simplify a few of these terms:

$$|\mathcal{T}|^2 = \frac{g^2}{4}\ \text{Tr}\left[-4(p_{1'}\cdot p_{2'}) - ms_{2'}\ \text{Tr}(\not{p}_{1'}\gamma_5\not{x}) - 4m^2 + ms_{2'}\ \text{Tr}(\gamma_5\not{x}\not{p}_{2'}) - ms_{1'}\ \text{Tr}(\gamma_5\not{x}\not{p}_{1'})\right.$$
$$\left.+s_{1'}s_{2'}\ \text{Tr}(\gamma_5\not{x}\not{p}_{1'}\gamma_5\not{x}\not{p}_{2'}) + ms_{1'}\ \text{Tr}(\gamma_5\not{x}\not{p}_{2'}) - m^2 s_{1'}s_{2'}\ \text{Tr}(\gamma_5\not{x}\gamma_5\not{x})\right]$$

Now we use the definition of s:

$$s = -(p_{1'} + p_{2'})^2 = -p_{1'}^2 - p_{2'}^2 - 2p_{1'}\cdot p_{2'}$$

Solving this, and recalling that $p_{1'}^2 = -m^2$, we have $p_{1'}\cdot p_{2'} = m^2 - \frac{s}{2}$.

Also we recall that $\{\gamma^5, \gamma^\mu\} = 0$. Then:

$$|\mathcal{T}|^2 = \frac{g^2}{4}\left\{4\left(\frac{s-2m^2}{2}\right) - m(s_{1'} + s_{2'})\ \text{Tr}(\not{p}_1\gamma_5\not{x}) + m(s_{1'} + s_{2'})\ \text{Tr}(\gamma_5\not{x}\not{p}_2)\right.$$
$$\left.+m^2 s_{1'}s_{2'}\ \text{Tr}(\gamma_5\gamma_5\not{x}\not{x}) + s_{1'}s_{2'}\ \text{Tr}(\gamma_5\gamma_5\not{x}\not{p}_{1'}\not{x}\not{p}_{2'}) - 4m^2\right\}$$

Now we use 28.16 to drop some of these terms. Further, $\gamma^5\gamma^5 = I$. Thus:

$$|\mathcal{T}|^2 = \frac{g^2}{4}\left\{2s - 4m^2 - 4m^2 s_{1'}s_{2'}(x\cdot x) + s_{1'}s_{2'}\ \text{Tr}(\not{x}\not{p}_{1'}\not{x}\not{p}_{2'}) - 4m^2\right\}$$

We further use equation 47.13:

$$|\mathcal{T}|^2 = g^2\left\{\frac{s}{2} - 2m^2 - m^2(x\cdot x)s_{1'}s_{2'} + \left[(x\cdot p_{2'})(x\cdot p_{1'}) - (x\cdot x)(p_{1'}\cdot p_{2'}) + (x\cdot p_{1'})(x\cdot p_{2'})\right]s_{1'}s_{2'}\right\} \tag{48.4.5}$$

Now we recall that $x = (0, \hat{x})$, so $x\cdot x = 1$:

$$|\mathcal{T}|^2 = g^2\left\{\frac{s}{2} - 2m^2 - m^2 s_{1'}s_{2'} + \left[(x\cdot p_{2'})(x\cdot p_{1'}) - (x\cdot x)(p_{1'}\cdot p_{2'}) + (x\cdot p_{1'})(x\cdot p_{2'})\right]s_{1'}s_{2'}\right\}$$

Now the three-momentum is in the z-direction, so $p_i\cdot x = 0$. Then:

$$|\mathcal{T}|^2 = g^2\left\{\frac{s}{2} - 2m^2 - m^2 s_{1'}s_{2'} - (p_{1'}\cdot p_{2'})s_{1'}s_{2'}\right\}$$

Using our definition of s again:

$$|\mathcal{T}|^2 = g^2\left\{\frac{s}{2} - 2m^2 - m^2 s_{1'}s_{2'} + \left(\frac{s}{2} - m^2\right)s_{1'}s_{2'}\right\}$$

For decay, $s = M^2$. Thus:

$$|\mathcal{T}|^2 = \frac{g^2}{2}\left\{M^2(1+s_{1'}s_{2'}) - 4m^2(1+s_{1'}s_{2'})\right\}$$

Simplifying:

$$\boxed{|\mathcal{T}|^2 = \frac{g^2}{2}(M^2-4m^2)(1+s_{1'}s_{2'})}$$

Now, the Lagrangian should be even under parity. Our interaction term is $\mathcal{L}_1 = g\phi\overline{\Psi}\Psi$. $\overline{\Psi}\Psi$ is even by 40.37, so ϕ must be even as well. In other words, the initial parity is +1.

We work in the center of mass frame, where the scalar has no motion and therefore no orbital angular momentum. The scalar is spin-0, so there is no spin angular momentum. Thus, the initial angular momentum must be zero.

The parity after decay is given by 40.17: $\mathcal{P} = (-1)^{\ell+1}$. Now imagine that $M = 2m$: in this case there can be no motion relative to the original scalar after decay. As a result, $\ell = 0$, and the parity afterwords must be -1. This is no good, as the parity of the original scalar was +1. So, we expect $|\mathcal{T}|^2 = 0$ if $M = 2m$, as is indeed the case.

Now consider the final angular momentum. The orbital angular momentum is nonzero, because the spin and the linear momentum are not collinear. Thus, we need the spin angular momentum to be nonzero in order to cancel this out. Therefore, we need the spins to be in the same direction (if they are in the opposite direction, they will cancel each other and cannot cancel the orbital angular momentum). Thus, we expect $|\mathcal{T}|^2 = 0$ if the spins are in the opposite directions, as is indeed the case.

(c) Compute $|\mathcal{T}|^2$ for decay into an electron with helicity s_1 and a positron with helicity s_2. You should find thta the decay rate is zero if $s_1 = s_2$. Discuss this in light of conservation of angular momentum and of parity.

This is the same problem as before, except this time we have helicity rather than spin specified, meaning that the spin must be along the same axis as the three-momentum (we'll use the z-axis). Thus, our answer from part (b) holds up to equation (48.4.5) with $x \to z$. We must also be much clearer about the helicity axes (since the helicities are not necessarily in the same direction as each other): let's call them z_1 and z_2 to avoid confusion. Thus:

$$|\mathcal{T}|^2 = g^2\left\{\frac{s}{2} - 2m^2 + \left[(z_1\cdot p_{2'})(z_2\cdot p_{1'}) - (z_1\cdot z_2)(p_{1'}\cdot p_{2'}) + (z_1\cdot p_{1'})(z_2\cdot p_{2'}) - m^2(z_1\cdot z_2)\right] s_{1'}s_{2'}\right\}$$

Next, we note the kinematics of the collision. In the center of mass frame, we originally had a scalar with four-momentum given by $p_1 = (M, 0)$, which has magnitude $-M^2$. After the collision, we had two fermions with $p_{1'} = (E, \vec{p})$ and $p_{2'} = (E, -\vec{p})$. Adding these, we get $p_1 + p_2 = (2E, 0)$, which has magnitude $-4E^2$. Equating these two magnitudes, we determine that $E = \frac{M}{2}$.

Next, we consider that $-m^2 = p^2 = -E^2 + |\vec{p}|^2 = -\frac{M^2}{4} + |\vec{p}|^2$, which implies that $|\vec{p}|^2 = \frac{M^2}{4} - m^2$, and so $|\vec{p}| = \frac{\sqrt{M^2-4m^2}}{2}$.

The result of this is that $p_1 = \left(\frac{M}{2}, \frac{\sqrt{M^2-4m^2}}{2}\hat{z}\right)$ and $p_2 = \left(\frac{M}{2}, -\frac{\sqrt{M^2-4m^2}}{2}\hat{z}\right)$.

What about z_1 and z_2? We recall our conventions (from page 241) that $z^2 = 1$ and $p \cdot z = 0$. Thus:

$$z_1 = \left(\frac{\sqrt{M^2 - 4m^2}}{2m}, \frac{M}{2m}\hat{z}\right)$$

$$z_2 = \left(\frac{\sqrt{M^2 - 4m^2}}{2m}, -\frac{M}{2m}\hat{z}\right)$$

Our expression is (using $s = M^2$):

$$|\mathcal{T}|^2 = g^2 \left\{ \frac{M^2}{2} - 2m^2 + \left[(z_1 \cdot p_{2'})(z_2 \cdot p_{1'}) - (z_1 \cdot z_2)(p_{1'} \cdot p_{2'}) + (z_1 \cdot p_{1'})(z_2 \cdot p_{2'}) - m^2(z_1 \cdot z_2)\right] s_{1'} s_{2'} \right\}$$

$p_1 \cdot z_1 = 0$ of course, so:

$$|\mathcal{T}|^2 = g^2 \left\{ \frac{M^2}{2} - 2m^2 + \left[(z_1 \cdot p_{2'})(z_2 \cdot p_{1'}) - (z_1 \cdot z_2)(p_{1'} \cdot p_{2'}) - m^2(z_1 \cdot z_2)\right] s_{1'} s_{2'} \right\}$$

Using our expressions for p_i and z, we have:

$$|\mathcal{T}|^2 = g^2 \left\{ \frac{M^2}{2} - 2m^2 + \left[\frac{M^2(M^2 - 4m^2)}{4m^2} - (z_1 \cdot z_2)(p_{1'} \cdot p_{2'} + m^2(z_1 \cdot z_2))\right] s_{1'} s_{2'} \right\}$$

Using our expressions for p_i and z again, we have:

$$|\mathcal{T}|^2 = g^2 \left\{ \frac{M^2}{2} - 2m^2 + \left[\frac{M^2(M^2 - 4m^2)}{4m^2} - \left(1 - \frac{M^2}{2m^2}\right)\left(-\frac{M^2}{2} + m^2 + m^2\right)\right] s_{1'} s_{2'} \right\}$$

Simplifying the part in brackets gives:

$$|\mathcal{T}|^2 = g^2 \left\{ \frac{M^2}{2} - 2m^2 + \left(\frac{M^2}{2} - 2m^2\right) s_{1'} s_{2'} \right\}$$

Factoring:

$$|\mathcal{T}|^2 = \frac{g^2}{2}\left(M^2 - 4m^2\right)(1 + s_{1'} s_{2'})$$

The initial angular momentum is zero as before. The spins are parallel to the 3-momentum (since we are in a state of definite chirality), so no orbital angular momentum in the final state is possible. Thus, the final spin angular momentum must also be zero.

What is the final spin angular momentum? We have $s_{1'}$ along the direction of the three-momentum, and $s_{2'}$ along the direction of the 2-particle's three momentum, which we know is (in the center of mass frame) equal and opposite to the one-particle's three momentum. Thus, the total spin is $s_1 - s_2$. For this to go to zero, we therefore need $s_1 = s_2$. For any other possible arrangement (and in this case there is only one other possible arrangement, $s_1 = -s_2$), angular momentum will not be conserved, and so the transitition amplitude must vanish. This is exactly what we see.

(d) Now consider changing the interaction to $\mathcal{L} = ig\phi\overline{\Psi}\gamma_5\Psi$, and compute the spin-summed decay rate. Explain (in light of consevation of angular momentum and of parity) why the decay rate is larger than it was without the $i\gamma^5$ in the interaction.

We start with equation (48.4.1), this time making the substitution $g \to ig\gamma_5$. Then:

$$|\mathcal{T}|^2 = -g^2 \text{ Tr}\left[u_{s_{1'}}(\vec{p}_{1'})\overline{u}_{s_1'}(\vec{p}_{1'})\gamma_5 v_{s_{2'}}(\vec{p}_{2'})\overline{v}_{s_{2'}}(\vec{p}_{2'})\gamma_5\right]$$

Now we want to sum over the final states:

$$\langle|\mathcal{T}|^2\rangle = -g^2 \sum_{s_{1'}}\sum_{s_{2'}} \text{ Tr}\left[u_{s_{1'}}(\vec{p}_{1'})\overline{u}_{s_1'}(\vec{p}_{1'})\gamma_5 v_{s_{2'}}(\vec{p}_{2'})\overline{v}_{s_{2'}}(\vec{p}_{2'})\gamma_5\right]$$

Now we use 46.8 and 46.13:

$$\langle|\mathcal{T}|^2\rangle = -g^2 \text{ Tr}\left[(-\not{p}_1 + m)\gamma_5(-\not{p}_2 - m)\gamma_5\right]$$

We can anticommute the last γ^5 through the $\not{p}_2$, with the result:

$$\langle|\mathcal{T}|^2\rangle = -g^2 \text{ Tr}\left[(-\not{p}_1 + m)\gamma_5\gamma_5(\not{p}_2 - m)\right]$$

Now of course $\gamma_5^2 = 1$; hence:

$$\langle|\mathcal{T}|^2\rangle = -g^2 \text{ Tr}\left[(-\not{p}_1 + m)(\not{p}_2 - m)\right]$$

Multiplying the binomials, and dropping terms with an odd number of gamma matrices:

$$\langle|\mathcal{T}|^2\rangle = -g^2 \text{ Tr}\left[-\not{p}_1\not{p}_2 - m^2\right]$$

Taking the trace:

$$\langle|\mathcal{T}|^2\rangle = g^2\left[-4(p_1\cdot p_2) + 4m^2\right]$$

Using equation (48.4.2):

$$\langle|\mathcal{T}|^2\rangle = g^2\left[-4\left(m^2 - \frac{s}{2}\right) + 4m^2\right]$$

Now take $s = M^2$:

$$\langle|\mathcal{T}|^2\rangle = g^2\left[-4m^2 + 2M^2 + 4m^2\right]$$

which is:

$$\langle|\mathcal{T}|^2\rangle = 2g^2M^2$$

Now we compare this to equation (48.4.3). This is larger by a factor of $\frac{M^2}{M^2-4m^2}$. Recall that according to 11.48, the decay rate is directly proportional to $|\mathcal{T}|^2$. Thus, our decay rate is also larger by the same factor. For completeness, the decay rate of the new process is given by:

$$\Gamma = \frac{g^2 M}{8\pi}\sqrt{1 - \left(\frac{2m}{M}\right)^2}$$

Why is this rate larger? Why is the new interaction more prone to decay? In this interaction, equation 40.37 tells us that $\overline{\Psi} i\gamma_5 \Psi$ has odd parity; since the Lagrangian must have even parity, it follows that ϕ must have odd parity. Therefore, the parity in the initial state is odd.

In the final state, the parity is again given by $\mathcal{P} = (-1)^{\ell+1}$. Thus, ℓ must be even.

What about angular momentum? There is no angular momentum in the initial state, so the orbital angular momentum must be equal and opposite the spin angular momentum. The spin angular momentum is at most one (both fermions in the same direction). Therefore, ℓ is at most one. Since ℓ is even, it follows that $\ell = 0$.

Now in part (a), we had the requirement that there must be orbital angular momentum, otherwise there would be no way to conserve parity. As a result, we had an amplitude that got very small as the electron three-momentum decreased. Here in part (d) there is no such requirement, hence the matrix element, and the decay rate, are larger.

(e) Repeat parts (b) and (c) for the new form of the interaction, and explain any differences in the results.

We begin with equation (48.4.4), inserting the γ_5 matrices as needed.

$$|\mathcal{T}|^2 = -\frac{g^2}{4}\ \mathrm{Tr}\left[(1 - s_1\gamma_5\not{x})(-\not{p}_{1'} + m)\gamma_5(1 - s_2\gamma_5\not{x})(-\not{p}_{2'} - m)\gamma_5\right]$$

Now we do the multiplication, dropping those terms with an odd number of gamma matrices:

$$|\mathcal{T}|^2 = -\frac{g^2}{4}\ \mathrm{Tr}\Big[(-\not{p}_{1'})\gamma_5(-\not{p}_{2'})\gamma_5 + (-\not{p}_{1'})\gamma_5(-s_2\gamma_5\not{x})(-m)\gamma_5 + m\gamma_5(-m)\gamma_5$$

$$m\gamma_5(-s_2\gamma_5\not{x})(-\not{p}_2)\gamma_5 + (-s_1\gamma_5\not{x})(-\not{p}_{1'})\gamma_5(-m)\gamma_5 + (-s_1\gamma_5\not{x})(-\not{p}_{1'})\gamma_5(-s_2\gamma_5\not{x})(-\not{p}_{2'})\gamma_5$$

$$(-s_1\gamma_5\not{x})(m)\gamma_5(-\not{p}_{2'})\gamma_5 + (-s_1\gamma_5\not{x})(m)\gamma_5(-s_2\gamma_5\not{x})(-m)\gamma_5\Big]$$

Now we use $\{\gamma_5, \gamma^\mu\} = 0$, and $\gamma_5^2 = 1$. This gives:

$$|\mathcal{T}|^2 = -\frac{g^2}{4}\ \mathrm{Tr}\Big[-\not{p}_{1'}\not{p}_{2'} - ms_{2'}\not{p}_{1'}\not{x}\gamma_5 - m^2 + ms_2\not{x}\not{p}_{2'}\gamma_5 - ms_1\gamma_5\not{x}\not{p}_{1'} + s_1s_2\not{x}\not{p}_{1'}\not{x}\not{p}_{2'}$$

$$-ms_1\gamma_5\not{x}\not{p}_{2'} - m^2 s_1 s_2\not{x}\not{x}\Big]$$

Evaluating these traces, we have:

$$|\mathcal{T}|^2 = -\frac{g^2}{4}\left\{4(p_{1'}\cdot p_{2'} - 4m^2 + 4s_1s_2\left[(x\cdot p_{2'})(x\cdot p_{1'}) - (x\cdot x)(p_{1'}\cdot p_{2'}) + (x\cdot p_{1'})(x\cdot p_{2'})\right]\right.$$

$$+4m^2 s_1 s_2 (x \cdot x)\Big\}$$

Simplifying:

$$|\mathcal{T}|^2 = -g^2 \left\{(p_{1'} \cdot p_{2'}) - m^2 + s_1 s_2 \left[2(x \cdot p_{1'})(x \cdot p_{2'}) + (x \cdot x)(m^2 - (p_{1'} \cdot p_{2'}))\right]\right\}$$

Now we use $(x \cdot x) = 1$ and $p_{1'} \cdot x = 0$. Then:

$$|\mathcal{T}|^2 = -g^2 \left\{(p_{1'} \cdot p_{2'}) - m^2 + s_1 s_2 \left[m^2 - (p_{1'} \cdot p_{2'})\right]\right\}$$

Simplifying:

$$|\mathcal{T}|^2 = -g^2 \left\{(p_{1'} \cdot p_{2'}) - m^2 - s_1 s_2 \left[(p_{1'} \cdot p_{2'}) - m^2\right]\right\} \tag{48.4.6}$$

Factoring:

$$|\mathcal{T}|^2 = -g^2 \left(p_{1'} \cdot p_{2'} - m^2\right)(1 - s_1 s_2)$$

Recall that $p_1 \cdot p_2 = m^2 - \frac{M^2}{2}$. Then:

$$|\mathcal{T}|^2 = \frac{g^2}{2} M^2 (1 - s_1 s_2)$$

This is different from our result in part (b). Notice:

- Initially, we have no angular momentum and odd parity. Finally, we have (as discussed in part (d)), no orbital angular momentum. Thus, there must be no spin angular momentum, and so the spins must be equal and opposite. Thus, we expect that $|\mathcal{T}|^2$ will vanish if the spins are the same, which is what we do observe.
- The magnitude is different, as observed in part (d).
- There is no dependence on m! This might seem surprising; we would expect the "physics" to depend on both masses. But it still does – $|\mathcal{T}|^2$ is just a scattering amplitude, not an observable. The m dependence will enter when we calculate an observable (like Γ).

Now to repeat part (c) for the new interaction. As in the original part (c), we take equation (48.4.6) and change $x \to z_1, z_2$. We recall that $p_1 \cdot z_1 = 0$. Thus:

$$|\mathcal{T}|^2 = -g^2 \left\{(p_{1'} \cdot p_{2'}) - m^2 + s_1 s_2 \left[(z_1 \cdot p_2)(z_2 \cdot p_1) + (z_1 \cdot z_2)(m^2 - p_{1'} \cdot p_{2'})\right]\right\}$$

We defined z and p in part (c); we simply plug in and the result is:

$$|\mathcal{T}|^2 = -\frac{g^2}{2} M^2 (1 + s_1 s_2)$$

There is no difference from the result in part (c) except the magnitude, as discussed above.

Srednicki 48.5. The *charged pion* π^- is represente by a complex scalar field ϕ, the *muon* μ^- by a Dirac field $\mathcal{M}$, and the *muon neutrino* ν_μ by a spin-projected Dirac field $P_L \mathcal{N}$, where $P_L = \frac{1}{2}(1 - \gamma_5)$. The charged pion can decay to a muon and a muon antineutrino via the interaction:

$$\mathcal{L}_1 = 2 c_1 G_F f_\pi \partial_\mu \phi \overline{\mathcal{M}} \gamma^\mu P_L \mathcal{N} + \text{h.c.}$$

where c_1 is the cosine of the *Cabibbo angle*, G_F is the *Fermi constant*, and f_π is the *pion decay constant*.

(a) Compute the charged pion decay rate Γ.

Our interaction Lagrangian, written in full, is:

$$\mathcal{L}_1 = 2c_1 G_F f_\pi \partial_\mu \phi \overline{\mathcal{M}} \gamma^\mu \frac{1}{2}(1-\gamma_5)\mathcal{N}$$

Note that we have neglected the Hermitian conjugate, because we have a μ^- decaying, not a μ^+.

The Feynman diagram is the same as in the previous problem, so the only new complication is the vertex factor. Recall that we replace the derivative with ik, drop the field terms, and add a factor of i. Then:

$$\text{V.F.} = ic_1 G_F f_\pi (ik_{1\mu})\gamma^\mu(1-\gamma_5)$$

Thus, the magnitude of the diagram is:

$$i\mathcal{T} = \overline{u}_{s_{1'}}(\vec{p}_{1'})\left[ic_1 G_F f_\pi (ik_{1\mu})\gamma^\mu(1-\gamma_5)\right] v_{s_{2'}}(\vec{p}_{2'})$$

Define $g = c_1 G_F f_\pi$

$$\mathcal{T} = ig\overline{u}_{s_{1'}}(\vec{p}_{1'})\left[(\not{k}_1)(1-\gamma_5)\right] v_{s_{2'}}(\vec{p}_{2'})$$

Now we want to simplify this using the Dirac Equation, otherwise we get horrible messes of γ matrices. Let's start by recognizing that $k_1 = p_{1'} + p_{2'}$, so:

$$\mathcal{T} = ig\overline{u}_{s_{1'}}(\vec{p}_{1'})\left[(\not{p}_{1'} + \not{p}_{2'})(1-\gamma_5)\right] v_{s_{2'}}(\vec{p}_{2'})$$

$p_{1'}$ represents the muon, so we should have it act on the $\overline{u}$ according to equation 38.16. Thus:

$$\mathcal{T} = ig\overline{u}_{s_{1'}}(\vec{p}_{1'})\left[(-m_\mu + \not{p}_{2'})(1-\gamma_5)\right] v_{s_{2'}}(\vec{p}_{2'})$$

$p_{2'}$ represents the neutrino, so we should have it act on the v. First, we have to (anti)commute through the $(1-\gamma_5)$:

$$\mathcal{T} = ig\overline{u}_{s_{1'}}(\vec{p}_{1'})\left[-m_\mu + \not{p}_{2'} + m_\mu\gamma_5 - \not{p}_{2'}\gamma_5\right] v_{s_{2'}}(\vec{p}_{2'})$$

which is:

$$\mathcal{T} = ig\overline{u}_{s_{1'}}(\vec{p}_{1'})\left[-m_\mu + \not{p}_{2'} + m_\mu\gamma_5 + \gamma_5\not{p}_{2'}\right] v_{s_{2'}}(\vec{p}_{2'})$$

Now we use equation 38.1, but we can neglect the neutrino mass. Thus:

$$\mathcal{T} = -igm_\mu\overline{u}_{s_{1'}}(\vec{p}_{1'})\,(1-\gamma_5)\,v_{s_{2'}}(\vec{p}_{2'})$$

Taking the conjugate of this (recalling that $\overline{i\gamma_5} = i\gamma_5$):

$$\overline{\mathcal{T}} = igm_\mu\overline{v}_{s_{2'}}(\vec{p}_{2'})\,(1+\gamma_5)\,u_{s_{1'}}(\vec{p}_{1'})$$

Thus:

$$|\mathcal{T}|^2 = g^2 m_\mu^2 \overline{u}_{s_{1'}}(\vec{p}_{1'})\,(1-\gamma_5)\,v_{s_{2'}}(\vec{p}_{2'})\overline{v}_{s_{2'}}(\vec{p}_{2'})\,(1+\gamma_5)\,u_{s_{1'}}(\vec{p}_{1'})$$

Next we use our usual trick of writing in index notation, reordering, then dropping the index notation; the result is:

$$|\mathcal{T}|^2 = g^2 m_\mu^2 \text{ Tr}\left[u_{s_{1'}}(\vec{p}_{1'})\overline{u}_{s_{1'}}(\vec{p}_{1'})\,(1-\gamma_5)\,v_{s_{2'}}(\vec{p}_{2'})\overline{v}_{s_{2'}}(\vec{p}_{2'})\,(1+\gamma_5)\right]$$

Now we can sum over the final states, and use 46.8 and 46.16 (again neglecting the neutrino mass):

$$|\mathcal{T}|^2 = g^2 m_\mu^2 \text{ Tr}\left[(-\not{p}_{1'} + m_\mu)\,(1-\gamma_5)\,(-\not{p}_{2'})\,(1+\gamma_5)\right]$$

Moving $\not{p}_{2'}$ to the left:

$$|\mathcal{T}|^2 = g^2 m_\mu^2 \text{ Tr}\left[(-\not{p}_{1'} + m_\mu)(-\not{p}_{2'})\,(1+\gamma_5)\,(1+\gamma_5)\right]$$

Multiplying these last two binomials, we have:

$$|\mathcal{T}|^2 = 2g^2 m_\mu^2 \text{ Tr}\left[(-\not{p}_{1'} + m_\mu)(-\not{p}_{2'})(1+\gamma_5)\right]$$

Performing this multiplication, only one term survives:

$$|\mathcal{T}|^2 = 2g^2 m_\mu^2 \text{ Tr}\left[\not{p}_{1'}\not{p}_{2'}\right]$$

which is:

$$|\mathcal{T}|^2 = -8g^2 m_\mu^2 (p_1 \cdot p_2)$$

Solving for the Mandelstam variable as usual, we have:

$$p_1 \cdot p_2 = \frac{m_\mu^2 - m_\pi^2}{2}$$

Thus:

$$|\mathcal{T}|^2 = 4g^2 m_\mu^2 (m_\pi^2 - m_\mu^2)$$

Now we put this into 11.48:

$$d\Gamma = \frac{1}{2m_\pi} 4m_\mu^2 g^2 (m_\pi^2 - m_\mu^2) \frac{|k_{1'}|}{16\pi^2 m_\pi} d\Omega_{cm}$$

11.3 gives, after simplification, that $|k_{1'}| = \frac{1}{2m_\pi}(m_\pi^2 - m_\mu^2)$. Thus:

$$d\Gamma = \frac{g^2}{16\pi^2} \frac{(m_\pi^2 - m_\mu^2)^2 m_\mu^2}{m_\pi^3} d\Omega_{cm}$$

Integrating both sides, we have:

$$\Gamma = \frac{g^2}{4\pi} \frac{(m_\pi^2 - m_\mu^2)^2 m_\mu^2}{m_\pi^3}$$

(b) The charged pion mass is $m_\pi = 139.6$ MeV, the muon mass is $m_\mu = 105.7$ MeV, and the muon neutrino is massless. The Fermi constant is measured in muon decay to be $G_F = 1.166 \times 10^{-5}\text{GeV}^{-2}$, and the cosine of the Cabibbo angle is measured in nuclear beta decays to be $c_1 = 0.974$. The measured value of the charged pion lifetime is 2.603×10^{-8} s. Determine the value of the f_π in MeV. Your result is too large by 0.8%, due to neglect of electromagnetic loop corrections.

Plugging in numbers (have to convert, $G_F = 1.166 \times 10^{-11}\text{GeV}^{-2}$), we have:

$$\Gamma = 2.91 \times 10^{-18} f_\pi^2 \text{ MeV}^{-1}$$

Now $\tau = 1/\Gamma$, so:

$$\tau = \frac{3.43 \times 10^{17}}{f_\pi^2} \text{ MeV}$$

Now let's multiply by Planck's constant to bring some units of time into this. This gives:

$$\tau = \frac{2.258 \times 10^{-4}}{f_\pi^2} \text{ MeV}^2 \text{ s}$$

Equating with the measured value of τ, we have:

$$2.603 \times 10^{-8} \text{ s} = \frac{2.258 \times 10^{-4}}{f_\pi^2} \text{ MeV}^2 \text{ s}$$

Solving this, we have

$$f_\pi = 93.13 \text{ MeV}$$

Note: we actually calculated G_F back in problem 11.3; the only non-contained part of this problem is the Cabbibo angle, which we'll discuss later on. Though of course, the more fundamental question is where these Lagrangians come from – we'll get there also.