Srednicki Chapter 47

QFT Problems & Solutions

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Srednicki 47.1. Verify equation 47.16.

Recall that $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. We choose to write this in a peculiar way:

$$\gamma^5 = i\varepsilon_{\mu\nu\sigma\rho}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho \text{ (no summation)}$$

Now we can multiply by $\gamma^\mu\gamma^\nu$. It's fine that these are doubly defined: all four gamma matrices are already present, so we really are adding in two of the same ones again. The only complication is that μ and ν are required by the Levi-Cevita symbol to be different; as a result, we'll consider the case where $\mu = \nu$ separately. Thus:

$$\gamma^5\gamma^\mu\gamma^\nu = i\varepsilon_{\mu\nu\sigma\rho}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho\gamma^\mu\gamma^\nu \text{ (no summation)}, \ \mu \neq \nu$$

Now the Levi-Cevita symbol requires $\mu \neq \nu \neq \sigma \neq \rho$ – if this is not true, then we get zero and we're done. Recall further that the gamma matrices anticommute with each other. Then:

$$\gamma^5\gamma^\mu\gamma^\nu = -i\varepsilon_{\mu\nu\sigma\rho}\gamma^\mu\gamma^\mu\gamma^\nu\gamma^\nu\gamma^\sigma\gamma^\rho \text{ (no summation)}, \ \mu \neq \nu$$

Now the form of the γ matrix (equation 36.7) shows that $(\gamma^0)^2 = I$, $(\gamma^i)^2 = -I$. Then:

$$\gamma^5\gamma^\mu\gamma^\nu = \pm i\varepsilon_{\mu\nu\sigma\rho}\gamma^\sigma\gamma^\rho \text{ (no summation)}, \ \mu \neq \nu$$

Next we take the trace, the constants can go outside the trace:

$$Tr(\gamma^5\gamma^\mu\gamma^\nu) = \pm i\varepsilon_{\mu\nu\sigma\rho}Tr(\gamma^\sigma\gamma^\rho) \text{ (no summation)}, \ \mu \neq \nu$$

Now 47.8, with $\sigma \neq \rho$, gives:

$$Tr(\gamma^5\gamma^\mu\gamma^\nu) = 0$$

as expected.

It remains to consider the case where $\mu = \nu$. Then:

$$Tr(\gamma^5\gamma^\mu\gamma^\nu) = Tr\left(\gamma^5(\gamma^\mu)^2\right)$$

As discussed above:

$$Tr(\gamma^5\gamma^\mu\gamma^\nu) = \pm Tr\left(\gamma^5\right)$$

which by 47.15 is:

$$Tr(\gamma^5\gamma^\mu\gamma^\nu) = 0$$

Srednicki 47.2. Verify equations 47.20 and 47.20.

We have:

$$\gamma^\mu \not{a}\not{b}\gamma_\mu = a_\nu b_\sigma \gamma^\mu\gamma^\nu\gamma^\sigma\gamma_\mu$$

We anticommute:

$$\gamma^\mu \not{a}\not{b}\gamma_\mu = a_\nu b_\sigma \left(-2g^{\mu\nu} - \gamma^\nu\gamma^\mu\right)\gamma^\sigma\gamma_\mu$$

This is:

$$\gamma^\mu \not{a}\not{b}\gamma_\mu = -2g^{\mu\nu}a_\nu b_\sigma\gamma^\sigma\gamma_\mu - a_\nu b_\sigma\gamma^\nu\gamma^\mu\gamma^\sigma\gamma_\mu$$

Restoring the slash notation:

$$\gamma^\mu \not{a}\not{b}\gamma_\mu = -2\not{b}\not{a} - \not{a}\gamma^\mu\not{b}\gamma_\mu$$

Now we use equation 47.19:

$$\gamma^\mu \not{a}\not{b}\gamma_\mu = -2\not{b}\not{a} - \not{a}(d-2)\not{b}$$

and now equation 47.10:

$$\gamma^\mu \not{a}\not{b}\gamma_\mu = 2\not{a}\not{b} + 4ab - (d-2)\not{a}\not{b}$$

which is:

$$\gamma^\mu \not{a}\not{b}\gamma_\mu = 4ab - (d-4)\not{a}\not{b}$$

□

Next we have:

$$\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = a_\nu b_\sigma c_\rho \gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho\gamma_\mu$$

Anticommuting:

$$\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = a_\nu b_\sigma c_\rho \left(-\gamma^\nu\gamma^\mu - 2g^{\mu\nu}\right)\gamma^\sigma\gamma^\rho\gamma_\mu$$

This gives:

$$\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = -a_\nu b_\sigma c_\rho\gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho\gamma_\mu - 2a_\nu b_\sigma c_\rho g^{\mu\nu}\gamma^\sigma\gamma^\rho\gamma_\mu$$

Restoring the slash notation in the first term; anticommuting in the second term:

$$\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = -\not{a}\gamma^\mu\not{b}\not{c}\gamma_\mu - 2a_\nu b_\sigma c_\rho\left(-\gamma^\rho\gamma^\sigma - 2g^{\rho\sigma}\right)\gamma^\nu$$

Distribute in the second term; use equation 47.20 in the first term:

$$\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = -4\not{a}bc + (d-4)\not{a}\not{b}\not{c} + 2a_\nu b_\sigma c_\rho\gamma^\rho\gamma^\sigma\gamma^\nu + 4a_\nu b_\sigma c_\rho g^{\rho\sigma}\gamma^\nu$$

This gives:

$$\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = -4\not{a}bc + (d-4)\not{a}\not{b}\not{c} + 2\not{c}\not{b}\not{a} + 4bc\not{a}$$

bc is a Lorentz-invariant scalar, so we don't have to worry about commutation; the first and last terms cancel. Therefore:

$$\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = (d-4)\not{a}\not{b}\not{c} + 2\not{c}\not{b}\not{a}$$

□

Srednicki 47.3. Show that the most general 4×4 matrix can be written as a linear combination (with complex coefficients) of 1, γ^μ, $S^{\mu\nu}$, $\gamma^\mu\gamma_5$, and γ_5 where 1 is the identity matrix and $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$. Hint: if A and B are two different members of this set, prove linear independence by showing that Tr $A^\dagger B = 0$. Then count.

Let's start by showing that all the members of these groups are linearly independent. 1 and γ^5 only have one element, so that is trivial. Then we have the γ matrices:

$$Tr\left[(\gamma^\mu)^\dagger \gamma^\nu\right] = \pm Tr\left[\gamma^\mu \gamma^\nu\right] = 0$$

where the first equality is because γ^0 is Hermitian and γ^i are anti-Hermitian, and the second equality follows from 47.8.

Next we can do the $S^{\mu\nu}$s:

$$Tr\left[(S^{\mu\nu})^\dagger, S^{\rho\sigma}\right] \propto Tr\left[\left(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu\right)\left(\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho\right)\right]$$

The sign in the first term changed because of the dagger; the dagger will also perhaps change the sign, but this is absorbed by the proportionality symbol.

Now we have four terms with four gamma matrices each. If all four gamma matrices are different, then we have something proportional to $Tr(\gamma^5)$, which is zero by 47.15. If two of the gamma matrices are the same, then we can anticommute those (if necessary) until they are next to one another, at which point $(\gamma^\mu)^2 = \pm I$. We will be left with something proportional to $\gamma^\mu\gamma^\nu$, the trace of which is zero by equation 47.8. Finally, if all four gamma matrices are the same, then we have $[S^{\mu\mu}, S^{\mu\mu}] = [0, 0] = 0$, since S is antisymmetric.

And then we have to do $\gamma^\mu\gamma_5$. We have:

$$Tr\left[(\gamma^\mu)^\dagger \gamma_5\right] \propto Tr\left[\gamma^\mu \gamma^5\right] = 0$$

where the first equality follows by the Hermitian/anti-Hermitian nature of the γ matrices, and the second equality follows from equation 47.7.

Now we can do the counting:

- 1, 1 matrix.
- γ^μ, 4 linearly-independent matrices.
- $S^{\mu\nu}$, an anti-symmetric tensor. An anti-symmetric 4x4 tensor has 6 independent components; we showed above that these are linearly independent.
- $\gamma^\mu\gamma_5$, 4 linearly-independent matrices.
- γ_5, 1 matrix.

We therefore have 16 matrices, which is just the right number to express an arbitrary 4x4 matrix. However, it remains to show that these five groups are linearly independent from one another. 10 combinations to consider:

I.

$$Tr\left[1^{\dagger}\gamma^{\mu}\right] = Tr\left[\gamma^{\mu}\right] = 0$$

which follows from 36.39.

II.

$$Tr\left[1^{\dagger}S^{\mu\nu}\right] = Tr\left[S^{\mu\nu}\right] \propto Tr\begin{bmatrix} \sigma^{\rho} & 0 \\ 0 & \sigma^{\rho} \end{bmatrix} = 0$$

where the second proportionality follows from studying 36.51-36.53 for $\mu \neq \nu$, and the third equality follows because the Pauli matrices are traceless.

III.

$$Tr\left[1^{\dagger}\gamma^{\mu}\gamma_5\right] = Tr\left[\gamma^{\mu}\gamma_5\right] = 0$$

which follows from 47.7.

IV.

$$Tr\left[1^{\dagger}\gamma_5\right] = Tr\left[\gamma_5\right] = 0$$

which follows from 47.15.

V.

$$Tr\left[(\gamma^{\mu})^{\dagger}S^{\nu\rho}\right] \propto Tr\left[\gamma^{\mu}S^{\nu\rho}\right] \propto Tr\left[\gamma^{\mu}\right] = 0$$

where the first proportionality is due to the Hermitian or anti-Hermitian nature of γ^{μ}. The second proportionality is more complicated: we showed in case II that $S^{\mu}\nu$ is proportional to a Pauli matrix times the identity. The gamma matrix is a 2x2 matrix with a Pauli matrix on the off-diagonal terms. Multiplying these, we get something proportional to a gamma matrix, which has no trace.

VI.

$$Tr\left[(\gamma^{\mu})^{\dagger}\gamma^{\nu}\gamma_5\right] \propto Tr\left[\gamma^{\mu}\gamma^{\nu}\gamma_5\right] = 0$$

where the last equality follows from equation 47.16.

VII.

$$Tr\left[(\gamma^{\mu})^{\dagger}\gamma_5\right] \propto Tr\left[\gamma^{\mu}\gamma_5\right] = 0$$

where the last equality follows from equation 47.7.

VIII.

$$Tr\left[\gamma_5^{\dagger}(\gamma^{\mu})^{\dagger}S^{\nu\rho}\right] \propto Tr\left[\gamma_5\gamma^{\mu}S^{\nu\rho}\right] \propto Tr\left[\gamma_5\gamma^{\kappa}\right] = 0$$

where the second proportionality follows because we showed that $S^{\mu\nu} \propto \sigma^{\rho}I$, and a gamma matrix multiplied by this will give another gamma matrix. The last proportionality follows from 47.7.

IX.

$$Tr\left[\gamma_5^\dagger S^{\mu\nu}\right] \propto Tr\left[\gamma_5 S^{\mu\nu}\right] = 0$$

where the last equality follows from equation 47.8 (recall that $S^{\mu\nu}$ is the the difference of two products of two gamma matrices each).

X.

$$Tr\left[\gamma_5^\dagger \gamma^\mu \gamma_5\right] \propto Tr\left[\gamma_5 \gamma^\mu \gamma_5\right] = 0$$

which follows because γ^5 has four gamma matrices, and so equation 47.7 applies.