Srednicki Chapter 40

QFT Problems & Solutions

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Srednicki 40.1. Find the transformation properties of $\overline{\Psi}S^{\mu\nu}\Psi$ and $\overline{\Psi}iS^{\mu\nu}\gamma_5\Psi$ under P, T, and C. Verify that they are both even under CPT, as claimed. Do either or both vanish if Ψ is a Majorana field?

First, we have:

$$P^{-1}\overline{\Psi}S^{\mu\nu}\Psi P$$

Recall from equation 40.3 that $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$. Since these are all gamma matrices, equation 40.35 applies:

$$P^{-1}\overline{\Psi}S^{\mu\nu}\Psi P = \overline{\Psi}\beta S^{\mu\nu}\beta\Psi$$

which is:

$$P^{-1}\overline{\Psi}S^{\mu\nu}\Psi P = \frac{i}{4}\left(\overline{\Psi}\beta\gamma^\mu\gamma^\nu\beta\Psi - \overline{\Psi}\beta\gamma^\nu\gamma^\mu\beta\Psi\right) \tag{40.1.1}$$

Recall that $\beta = \gamma^0$. From equation 36.9, β anticommutes with γ^i. Obviously, β and γ^0 commute (since they are the same). Finally, $\beta^2 = I$. So:

$$P^{-1}\overline{\Psi}S^{\mu\nu}\Psi P = \begin{cases} \frac{i}{4}\left(\overline{\Psi}\gamma^\mu\gamma^\nu\Psi - \overline{\Psi}\gamma^\nu\gamma^\mu\Psi\right) & \text{if } \mu,\, \nu \in [1,3] \text{ or } \mu = \nu = 0 \\ -\frac{i}{4}\left(\overline{\Psi}\gamma^\mu\gamma^\nu\Psi - \overline{\Psi}\gamma^\nu\gamma^\mu\Psi\right\} & \text{otherwise} \end{cases}$$

$$P^{-1}\overline{\Psi}S^{\mu\nu}\Psi P = \begin{cases} \overline{\Psi}S^{\mu\nu}\Psi & \text{if } \mu,\, \nu \in [1,3] \text{ or } \mu = \nu = 0 \\ -\overline{\Psi}S^{\mu\nu}\Psi & \text{otherwise} \end{cases}$$

We can write this more succinctly as:

$$\boxed{P^{-1}\overline{\Psi}S^{\mu\nu}\Psi P = \mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma\overline{\Psi}S^{\rho\sigma}\Psi} \tag{40.1.2}$$

Now consider:

$$P^{-1}\overline{\Psi}iS^{\mu\nu}\gamma_5\Psi P$$

Recall that $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. γ_0 commutes with itself and anticommutes with the other three. Thus, in the analog to equation (40.1.1), we acquire an extra negative sign. Everything else is the same as before. Thus:

$$\boxed{P^{-1}\overline{\Psi}iS^{\mu\nu}\gamma_5\Psi P = -\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma\overline{\Psi}iS^{\rho\sigma}\gamma_5\Psi}$$

Next we'll do charge conjugation:

$$C^{-1}\overline{\Psi}S^{\mu\nu}\Psi C$$

As before, $S^{\mu\nu}$ consists of gamma matrices, so equation 40.45 applies:

$$C^{-1}\overline{\Psi}S^{\mu\nu}\Psi C = \frac{i}{4}\overline{\Psi}C^{-1}(\gamma^\mu\gamma^\nu)^T C\Psi - \frac{i}{4}\overline{\Psi}C^{-1}(\gamma^\nu\gamma^\mu)^T C\Psi$$

which is:

$$C^{-1}\overline{\Psi}S^{\mu\nu}\Psi C = \frac{i}{4}\overline{\Psi}C^{-1}\gamma^{\nu T}\gamma^{\mu T}C\Psi - \frac{i}{4}\overline{\Psi}C^{-1}\gamma^{\mu T}\gamma^{\nu T}C\Psi$$

Now we insert a few copies of the identity:

$$C^{-1}\overline{\Psi}S^{\mu\nu}\Psi C = \frac{i}{4}\overline{\Psi}C^{-1}\gamma^{\nu T}CC^{-1}\gamma^{\mu T}C\Psi - \frac{i}{4}\overline{\Psi}C^{-1}\gamma^{\mu T}CC^{-1}\gamma^{\nu T}C\Psi \tag{40.1.3}$$

and equation 40.46 applies:

$$C^{-1}\overline{\Psi}S^{\mu\nu}\Psi C = \frac{i}{4}\overline{\Psi}\gamma^\nu\gamma^\mu\Psi - \frac{i}{4}\overline{\Psi}\gamma^\mu\gamma^\nu\Psi$$

which is:

$$C^{-1}\overline{\Psi}S^{\mu\nu}\Psi C = \frac{i}{4}\overline{\Psi}[\gamma^\nu, \gamma^\mu]\Psi$$

and so:

$$\boxed{C^{-1}\overline{\Psi}S^{\mu\nu}\Psi C = -\overline{\Psi}S^{\mu\nu}\Psi}$$

As for $C^{-1}\overline{\Psi}iS^{\mu\nu}\gamma_5\Psi C$, the analog of equation (40.1.3) simply has an extra factor of $C^{-1}i\gamma_5 C = i\gamma_5$, and so there is no net difference. Thus:

$$\boxed{C^{-1}\overline{\Psi}iS^{\mu\nu}\gamma_5\Psi C = -\overline{\Psi}iS^{\mu\nu}\gamma_5\Psi} \tag{40.1.4}$$

Charge conjugation takes each particle to its antiparticle. Since a Majorana field is its own antiparticle, it must be equal to its charge conjugate. In both cases considered here, the field is odd under charge conjugation, and so must be equal to its negative, a condition that is obviously only possible if the field is zero. Thus,

$$\boxed{\text{both vanish if } \Psi \text{ is a Majorana field.}}$$

Finally, we have, using equation 40.39:

$$T^{-1}\overline{\Psi}S^{\mu\nu}\Psi T = \overline{\Psi}\gamma_5 C^{-1}S^{\mu\nu *}C\gamma_5\Psi$$

This is:

$$T^{-1}\overline{\Psi}S^{\mu\nu}\Psi T = \overline{\Psi}\gamma_5 C^{-1}(\gamma^\mu\gamma^\nu)^* C\gamma_5\Psi - \frac{i}{4}\overline{\Psi}\gamma_5 C^{-1}(\gamma^\nu\gamma^\mu)^* C\gamma_5\Psi$$

which is:

$$T^{-1}\overline{\Psi}S^{\mu\nu}\Psi T = \overline{\Psi}\gamma_5 C^{-1}\gamma^{\nu *}\gamma^{\mu *}C\gamma_5\Psi - \frac{i}{4}\overline{\Psi}\gamma_5 C^{-1}\gamma^{\mu *}\gamma^{\nu *}C\gamma_5\Psi$$

we insert the identity operator:

$$T^{-1}\overline{\Psi}S^{\mu\nu}\Psi T = \overline{\Psi}\gamma_5 C^{-1}\gamma^{\nu *}CC^{-1}\gamma^{\mu *}C\gamma_5\Psi - \frac{i}{4}\overline{\Psi}\gamma_5 C^{-1}\gamma^{\mu *}CC^{-1}\gamma^{\nu *}C\gamma_5\Psi$$

inserting another identity:

$$T^{-1}\overline{\Psi}S^{\mu\nu}\Psi T = \overline{\Psi}\gamma_5 C^{-1}\gamma^{\nu *}C\gamma_5\gamma_5 C^{-1}\gamma^{\mu *}C\gamma_5\Psi - \frac{i}{4}\overline{\Psi}\gamma_5 C^{-1}\gamma^{\mu *}C\gamma_5\gamma_5 C^{-1}\gamma^{\nu *}C\gamma_5\Psi \quad (40.1.5)$$

Using equation 40.40:

$$T^{-1}\overline{\Psi}S^{\mu\nu}\Psi T = \begin{cases} \frac{i}{4}\overline{\Psi}\gamma^\nu\gamma^\mu\Psi - \frac{i}{4}\overline{\Psi}\gamma^\mu\gamma^\nu\Psi & \text{if } \mu,\, \nu \in [1,3] \\ -\frac{i}{4}\overline{\Psi}\gamma^\nu\gamma^\mu\Psi + \frac{i}{4}\overline{\Psi}\gamma^\mu\gamma^\nu\Psi & \text{otherwise} \end{cases}$$

which is:

$$T^{-1}\overline{\Psi}S^{\mu\nu}\Psi T = \begin{cases} -\overline{\Psi}S^{\mu\nu}\Psi & \text{if } \mu,\, \nu \in [1,3] \\ \overline{\Psi}S^{\mu\nu}\Psi & \text{otherwise} \end{cases}$$

which we can rewrite as:

$$\boxed{T^{-1}\overline{\Psi}S^{\mu\nu}\Psi T = -\mathcal{T}^\mu_{\ \rho}\mathcal{T}^\nu_{\ \sigma}S^{\rho\sigma}} \quad (40.1.6)$$

As for $T^{-1}\overline{\Psi}iS^{\mu\nu}\gamma_5 T$, the analog of equation (40.1.5) simply has an extra factor of $\gamma_5 C^{-1}(i\gamma_5)^* C\gamma_5$, which contributes a minus sign. Thus:

$$\boxed{T^{-1}\overline{\Psi}iS^{\mu\nu}\gamma_5\Psi T = \mathcal{T}^\mu_{\ \rho}\mathcal{T}^\nu_{\ \sigma}S^{\rho\sigma}}$$

Putting all this together, we have:

$$(CPT)^{-1}\overline{\Psi}S^{\mu\nu}\Psi(CPT) = T^{-1}P^{-1}C^{-1}\overline{\Psi}S^{\mu\nu}\Psi CPT$$

Using equation (40.1.4), we have:

$$(CPT)^{-1}\overline{\Psi}S^{\mu\nu}\Psi(CPT) = -T^{-1}P^{-1}\overline{\Psi}S^{\mu\nu}\Psi PT$$

Using equation (40.1.2), we have:

$$(CPT)^{-1}\overline{\Psi}S^{\mu\nu}\Psi(CPT) = -\mathcal{P}^\mu_{\ \rho}\mathcal{P}^\nu_{\ \sigma}T^{-1}\overline{\Psi}S^{\rho\sigma}\Psi T$$

Using equation (40.1.6), we have:

$$(CPT)^{-1}\overline{\Psi}S^{\mu\nu}\Psi(CPT) = \mathcal{P}^\mu_{\ \lambda}\mathcal{P}^\nu_{\ \kappa}\mathcal{T}^\lambda_{\ \rho}\mathcal{T}^\kappa_{\ \sigma}\overline{\Psi}S^{\rho\sigma}\Psi$$

$\mathcal{P}\mathcal{T} = -I$. Thus:

$$\boxed{(CPT)^{-1}\overline{\Psi}S^{\mu\nu}\Psi(CPT) = \overline{\Psi}S^{\rho\sigma}\Psi}$$

which is the definition of even.

As for $(CPT)^{-1}\overline{\Psi}iS^{\mu\nu}\gamma_5\Psi(CPT)$, we have the same thing except for two minus signs, which cancel. Therefore this is even too, ie:

$$\boxed{(CPT)^{-1}\overline{\Psi}iS^{\mu\nu}\gamma_5\Psi(CPT) = \overline{\Psi}iS^{\rho\sigma}\gamma_5\Psi}$$