Srednicki Chapter 4

QFT Problems & Solutions

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Srednicki 4.1. Verify eq. 4.12. Verify its limit as m → 0.

$$[\phi^+(x),\phi^-(x')]_\mp = [\int \widetilde{dk} e^{ikx} a(\mathbf{k}), \int \widetilde{dk'} e^{-ik'x'} a^\dagger(\mathbf{k'})]_\mp$$

$$[\phi^+(x),\phi^-(x')]_\mp = \int \widetilde{dk}\widetilde{dk'} e^{i(kx-k'x')}[a(\mathbf{k}), a^\dagger(\mathbf{k'})]_\mp$$

Using equation 4.2:

$$[\phi^+(x),\phi^-(x')]_\mp = \int \widetilde{dk}\widetilde{dk'} e^{i(kx-k'x')}(2\pi)^3 2\omega\delta^3(\mathbf{k}-\mathbf{k'})$$

which is:

$$[\phi^+(x),\phi^-(x')]_\mp = \int \widetilde{dk} d^3k' e^{i(kx-k'x')}\delta^3(\mathbf{k}-\mathbf{k'})$$

Doing the k' integral, we're left with:

$$[\phi^+(x),\phi^-(x')]_\mp = \int \widetilde{dk} e^{ik(x-x')}$$

Now we take advantage of the comment in the text, and work in the frame where t = t'. Then,

$$[\phi^+(x),\phi^-(x')]_\mp = \int \frac{d^3k}{(2\pi)^3 2\omega} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x'})}$$

Now we decide to work in polar coordinates:

$$[\phi^+(x),\phi^-(x')]_\mp = \int \frac{dk d\theta d\phi}{(2\pi)^3 2\omega} k^2 sin(\theta) e^{ikrcos\theta}$$

Now we do the ϕ integral:

$$[\phi^+(x),\phi^-(x')]_\mp = \int \frac{dk d\theta}{(2\pi)^2 2\omega} k^2 sin(\theta) e^{ikrcos\theta}$$

Now we do the θ integral:

$$[\phi^+(x),\phi^-(x')]_\mp = \int \frac{dk}{(2\pi)^2 2\omega} k^2 \frac{2sin(kr)}{kr}$$

which becomes:

$$[\phi^+(x), \phi^-(x')]_\mp = \int \frac{dk}{(2\pi)^2} \frac{ksin(kr)}{\omega r}$$

Plugging in for ω:

$$[\phi^+(x), \phi^-(x')]_\mp = \frac{1}{r(2\pi)^2} \int dk \frac{ksin(kr)}{\sqrt{k^2+m^2}}$$

which becomes:

$$[\phi^+(x), \phi^-(x')]_\mp = \frac{1}{4\pi^2 r} \int_0^\infty dk\ sin(kr) \left[1 + \left(\frac{m}{k}\right)^2\right]^{-1/2}$$

Now we integrate by parts:

$$[\phi^+(x), \phi^-(x')]_\mp = \frac{m^2}{4\pi^2 r^2} \int_0^\infty dk\ cos(kr) \left[1 + \left(\frac{m}{k}\right)^2\right]^{-3/2} \frac{1}{k^3}$$

You might be concerned about our boundary term, which we appear to have forgotten. In fact, it is customary to assume our functions are well-behaved at the arbitrarily high values, since they are unphysical. Then,

$$[\phi^+(x), \phi^-(x')]_\mp = \frac{m^2}{4\pi^2 r^2} \int_0^\infty dk\ cos(kr) \left[k^2 + m^2\right]^{-3/2}$$

Now let $t = \frac{k}{m} \implies k = tm$:

$$[\phi^+(x), \phi^-(x')]_\mp = \frac{m^3}{4\pi^2 r^2} \int_0^\infty dt\ cos(tmr) \left[(tm)^2 + m^2\right]^{-3/2}$$

These ms cancel:

$$[\phi^+(x), \phi^-(x')]_\mp = \frac{1}{4\pi^2 r^2} \int_0^\infty dt\ cos(tmr) \left[1 + t^2\right]^{-3/2}$$

which is:

$$[\phi^+(x), \phi^-(x')]_\mp = \frac{m}{4\pi^2 r} \frac{1}{mr} \int_0^\infty dt\ cos(tmr) \left[1 + t^2\right]^{-3/2}$$

This is a bessel function!

$$[\phi^+(x), \phi^-(x')]_\mp = \frac{m}{4\pi^2 r} K_1(mr)$$

which verifies equation 4.12. As for the asymptotic behavior, let's expand this around $m = 0$:

$$\lim_{m\to 0} [\phi^+(x), \phi^-(x')]_\mp = \frac{1}{4\pi^2 r^2} \left(1 + \frac{1}{4}(mr)^2 (2log(mr) + 2\gamma - 1 - log(4)) + \ldots\right)$$

These last three terms obviously vanish. The fourth-to-last term vanishes by l'Hôpital's Rule. Hence,

$$\lim_{m\to 0} [\phi^+(x), \phi^-(x')]_\mp = \frac{1}{4\pi^2 r^2}$$