Srednicki Chapter 39

QFT Problems & Solutions

A. George

March 22, 2013

Srednicki 39.1. Verify equation 39.24.

Let's start with the definition of the Noether Charge. We have:

$$Q = \int d^3x j^0$$

Note that we use Srednicki 22.9 rather than 22.27 since in this case, the symmetry being considered leaves the terms invariant, not merely invariant up to an overall divergence. Using the definition of the Noether current, Srednicki 22.6, we have:

$$Q = \int d^3x \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \delta\phi$$

The Lagrangian in question is equation 39.1, which can be written:

$$\mathcal{L} = i\overline{\Psi}\gamma^\mu \partial_\mu \Psi + m\overline{\Psi}\Psi$$

Then:

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = i\overline{\Psi}\gamma^0 \tag{39.1.1}$$

Further, the symmetry in question is:

$$\Psi \to e^{-i\alpha}\Psi = \Psi - i\alpha\Psi + \dots \tag{39.1.2}$$

Combining these numbered results, we have:

$$j^0 = i\overline{\Psi}\gamma^0(-i\alpha)\Psi$$

It is customary to remove the infinitesimal term form the Noether current. Thus:

$$j^0 = \overline{\Psi}\gamma^0\Psi$$

Which gives the Noether Charge as:

$$Q = \int d^3x \overline{\Psi}\gamma^0\Psi$$

Now we are ready to insert the solution to the Dirac Equation, Srednicki 37.30. Recall equation 37.17, the effect of which is that the expansion of $\overline{\Psi}$ is the same as Ψ with the daggers and exponentials "Hermitianated" and the spinors barred. Then:

$$Q = \int d^3x \left[\sum_{s=\pm} \int \widetilde{dp} \left(e^{-ipx} \overline{u}_s(p) b_s^\dagger(p) + e^{ipx} \overline{v}_s(p) d_s(p) \right) \right] \gamma^0$$

$$\left[\sum_{s'=\pm} \int \widetilde{dp'} \left(b_{s'}(p') u_{s'}(p') e^{ip'x} + d_{s'}^\dagger(p') v_{s'}(p') e^{-ip'x} \right) \right]$$

We distribute. Further, we recall that the operators act on vacuums or particles, not tensors, and therefore commute with the spinors.

$$Q = \sum_{s=\pm} \sum_{s'=\pm} \int d^3x \ \widetilde{dp} \ \widetilde{dp'} \left[e^{i(p'-p)x} \overline{u}_s(p) \gamma^0 u_{s'}(p') b_s^\dagger(p) b_{s'}(p') \right.$$

$$+ e^{-i(p+p')x} \overline{u}_s(p) \gamma^0 v_{s'}(p') d_{s'}^\dagger(p') b_s^\dagger(p) + e^{i(p+p')x} \overline{v}_s(p) \gamma^0 u_{s'}(p') d_s(p) b_{s'}(p')$$

$$\left. + \ e^{i(p-p')x} \overline{v}_s(p) \gamma^0 v_{s'}(p') d_s(p) d_s^\dagger(p') \right]$$

Next we use equation 38.21, killing the second sum as well as the second and third terms:

$$Q = 2\omega \sum_{s=\pm} \int d^3x \ \widetilde{dp} \ \widetilde{dp'} \left[e^{i(p'-p)x} b_s^\dagger(p) b_s(p') + e^{i(p-p')x} d_s(p) d_s^\dagger(p') \right]$$

Next we split up the exponentials into spatial and temporal parts. We also write the second integral explicitely:

$$Q = 2\omega \sum_{s=\pm} \int d^3x \ \widetilde{dp} \ \frac{d^3p'}{(2\pi)^3 2\omega} \left[e^{-i(\omega'-\omega)t} e^{i(\vec{p}'-\vec{p})\cdot\vec{x}} b_s^\dagger(p) b_s(p') + e^{-i(\omega-\omega')t} e^{i(\vec{p}-\vec{p}')\cdot\vec{x}} d_s(p) d_s^\dagger(p') \right]$$

Next we use equation 3.27:

$$Q = \sum_{s=\pm} \widetilde{dp} \ d^3p' \left[e^{-i(\omega'-\omega)t} \delta^3(\vec{p}' - \vec{p}) b_s^\dagger(p) b_s(p') + e^{-i(\omega-\omega')t} \delta^3(\vec{p} - \vec{p}') d_s(p) d_s^\dagger(p') \right]$$

We do the last integral. Note that the spatial parts are required to be equal, and the magnitudes are required to be equal (since the masses are the same), so the temporal parts must be equal too, ie $\omega = \omega'$. Thus:

$$Q = \sum_{s=\pm} \widetilde{dp} \left[b_s^\dagger(p) b_s(p) + d_s(p) d_s^\dagger(p) \right]$$

Finally, we use 39.17 to anticommute these terms:

$$Q = \sum_{s=\pm} \widetilde{dp} \left[b_s^\dagger(p) b_s(p) - d_s^\dagger(p) d_s(p) + (2\pi)^3 2\omega \right]$$

which is of course:

$$Q = \sum_{s=\pm} \widetilde{dp}\left[b_s^\dagger(p)b_s(p) - d_s^\dagger(p)d_s(p) + \text{const}\right]$$

as expected.

Srednicki 39.2. Use $[\Psi(x), M^{\mu\nu}] = -i(x^\mu\partial^\nu - x^\nu\partial^\mu)\Psi(x)$, plus whatever spinor identities you need, to show that

$$J_z b_s^\dagger(p\hat{z})|0\rangle = \frac{1}{2}s b_s^\dagger(p\hat{z})|0\rangle$$

$$J_z d_s^\dagger(p\hat{z})|0\rangle = \frac{1}{2}s d_s^\dagger(p\hat{z})|0\rangle$$

where $\vec{p} = p\hat{z}$ is the three-momentum, and $\hat{z}$ is a unit vector in the z direction.

Let's do both of these simultaneously. First, we use 39.10 and 39.12:

$$J_z b_s^\dagger(p\hat{z})|0\rangle = J_z \int d^3x\ e^{ipx}\overline{\Psi}(x)\gamma^0 u_s(p\hat{z})|0\rangle$$

$$J_z d_s^\dagger(p\hat{z})|0\rangle = J_z \int d^3x\ e^{ipx}\overline{v}_s(p\hat{z})\gamma^0\Psi(x)|0\rangle$$

Next we want to commute J_z through the matrix multiplication. J_z acting on a term $|n\rangle$ will give $s|n\rangle$. This will not be affected by any terms in front of the ket. In other words, it doesn't matter if J_z acts on the ket before the other stuff or after the other stuff – it therefore commutes with the other stuff.

The exception to this is if the "other stuff" changes the ket. The commutation between J_z and this "other stuff" needs to be carefully treated. Thus:

$$J_z b_s^\dagger(p\hat{z})|0\rangle = \int d^3x\ e^{ipx}[J_z, \overline{\Psi}(x)]\gamma^0 u_s(p\hat{z})|0\rangle$$

$$J_z d_s^\dagger(p\hat{z})|0\rangle = \int d^3x\ e^{ipx}\overline{v}_s(p\hat{z})\gamma^0[J_z, \Psi(x)]|0\rangle$$

Next, recall that $J_z = \frac{1}{2}\varepsilon_{zxy}M^{xy} = M^{xy}$, where the last equality follows because the Ms are antisymmetric. Thus:

$$J_z b_s^\dagger(p\hat{z})|0\rangle = \int d^3x\ e^{ipx}[M^{xy}, \overline{\Psi}(x)]\gamma^0 u_s(p\hat{z})|0\rangle$$

$$J_z d_s^\dagger(p\hat{z})|0\rangle = \int d^3x\ e^{ipx}\overline{v}_s(p\hat{z})\gamma^0[M^{xy}, \Psi(x)]|0\rangle$$

This second line allows us to use the identity given in the problem, but we need to adapt this identity for use in the first line (see my note in the slides if this is confusing):

$$[\Psi(x), M^{\mu\nu}] = -i(x^\mu\partial^\nu - x^\nu\partial^\mu)\Psi(x) + S^{\mu\nu}\Psi(x) \tag{39.2.1}$$

Barring, we have:

$$\overline{[\Psi(x), M^{\mu\nu}]} = \overline{-i(x^\mu\partial^\nu - x^\nu\partial^\mu)\Psi(x) + S^{\mu\nu}\Psi(x)}$$

$$[\overline{M}^{\mu\nu}, \overline{\Psi}(x)] = i(x^\mu\partial^\nu - x^\nu\partial^\mu)\overline{\Psi}(x) + \overline{\Psi(x)}\ \overline{S^{\mu\nu}}$$

M is Hermitian. Further, we use 38.15. Thus:

$$[M^{\mu\nu}, \overline{\Psi}(x)] = i(x^\mu\partial^\nu - x^\nu\partial^\mu)\overline{\Psi}(x) + \overline{\Psi}(x)S^{\mu\nu} \tag{39.2.2}$$

These commutators give:

$$J_z b_s^\dagger(p\hat{z})|0\rangle = \int d^3x\ e^{ipx}\left[i(x\partial^y - y\partial^x)\overline{\Psi}(x) + \overline{\Psi}(x)S^{xy}\right]\gamma^0 u_s(p\hat{z})|0\rangle$$

$$J_z d_s^\dagger(p\hat{z})|0\rangle = \int d^3x\ e^{ipx}\overline{v}_s(p\hat{z})\gamma^0\left[i(y\partial^x - x\partial^y)\Psi(x) - S^{xy}\Psi(x)\right]|0\rangle$$

The first (two) terms involve derivatives with respect to x or y. γ^0 is a constant and $\overline{u}(p\hat{z})$ are constant in space, and e^{ipz} is constant with respect to x or y. Integrating by parts then will shift the derivative to a constant term which vanishes. We assume that the boundary conditions are such that the remaining (surface) term from the integration by parts also vanishes. This gives:

$$J_z b_s^\dagger(p\hat{z})|0\rangle = \int d^3x\ e^{ipx}\overline{\Psi}(x)S^{xy}\gamma^0 u_s(p\hat{z})|0\rangle$$

$$J_z d_s^\dagger(p\hat{z})|0\rangle = -\int d^3x\ e^{ipx}\overline{v}_s(p\hat{z})\gamma^0 S^{xy}\Psi(x)|0\rangle$$

Next we recall:

$$S^{xy} = \frac{i}{4}[\gamma^x, \gamma^y]$$

$$S^{xy} = \frac{i}{4}\left[\begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}\right]$$

$$S^{xy} = \frac{i}{4}\begin{pmatrix} \sigma_y\sigma_x - \sigma_x\sigma_y & 0 \\ 0 & \sigma_y\sigma_x - \sigma_x\sigma_y \end{pmatrix}$$

$$S^{xy} = -\frac{i}{4}\begin{pmatrix} [\sigma_x, \sigma_y] & 0 \\ 0 & [\sigma_x, \sigma_y] \end{pmatrix}$$

$$S^{xy} = \frac{1}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}$$

This gives:

$$J_z b_s^\dagger(p\hat{z})|0\rangle = \frac{1}{2}\int d^3x\ e^{ipx}\overline{\Psi}(x)\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}\gamma^0 u_s(p\hat{z})|0\rangle$$

$$J_z d_s^\dagger(p\hat{z})|0\rangle = -\frac{1}{2}\int d^3x\ e^{ipx}\overline{v}_s(p\hat{z})\gamma^0\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}\Psi(x)|0\rangle$$

which is:

$$J_z b_s^\dagger(p\hat{z})|0\rangle = \frac{1}{2}\int d^3x\ e^{ipx}\overline{\Psi}(x)\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} u_s(p\hat{z})|0\rangle$$

$$J_z d_s^\dagger(p\hat{z})|0\rangle = -\frac{1}{2}\int d^3x\ e^{ipx}\overline{v}_s(p\hat{z})\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}\Psi(x)|0\rangle$$

These two matrices obviously commute. Hence:

$$J_z b_s^\dagger(p\hat{z})|0\rangle = \frac{1}{2}\int d^3x\ e^{ipx}\overline{\Psi}(x)\gamma^0\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} u_s(p\hat{z})|0\rangle$$

$$J_z d_s^\dagger(p\hat{z})|0\rangle = -\frac{1}{2}\int d^3x\ e^{ipx}\overline{v}_s(p\hat{z})\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}\gamma^0\Psi(x)|0\rangle$$

Next we use 38.12:

$$J_z b_s^\dagger(p\hat{z})|0\rangle = \frac{1}{2}\int d^3x\ e^{ipx}\overline{\Psi}(x)\gamma^0\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}\exp(i\eta p K^z) u_s(0)|0\rangle$$

$$J_z d_s^\dagger(p\hat{z})|0\rangle = -\frac{1}{2}\int d^3x\ e^{ipx}\overline{v}_s(0)\exp(i\eta p K^z)\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}\gamma^0\Psi(x)|0\rangle$$

What is K^z?

$$K^z = \frac{i}{2}\gamma^z\gamma^0 = \frac{i}{2}\begin{pmatrix} \sigma^z & 0 \\ 0 & -\sigma^z \end{pmatrix}$$

Thus:

$$J_z b_s^\dagger(p\hat{z})|0\rangle = \frac{1}{2}\int d^3x\ e^{ipx}\overline{\Psi}(x)\gamma^0\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}\exp\left[-\frac{1}{2}\eta p\begin{pmatrix} \sigma^z & 0 \\ 0 & -\sigma^z \end{pmatrix}\right] u_s(0)|0\rangle$$

$$J_z d_s^\dagger(p\hat{z})|0\rangle = -\frac{1}{2}\int d^3x\ e^{ipx}\overline{v}_s(0)\exp\left[-\frac{1}{2}p\begin{pmatrix} \sigma^z & 0 \\ 0 & -\sigma^z \end{pmatrix}\right]\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}\gamma^0\Psi(x)|0\rangle$$

It is clear that these two matrices commute. Thus:

$$J_z b_s^\dagger(p\hat{z})|0\rangle = \frac{1}{2}\int d^3x\ e^{ipx}\overline{\Psi}(x)\gamma^0\exp\left[-\frac{i}{2}\eta p\begin{pmatrix} \sigma^z & 0 \\ 0 & -\sigma^z \end{pmatrix}\right]\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} u_s(0)|0\rangle$$

$$J_z d_s^\dagger(p\hat{z})|0\rangle = -\frac{1}{2}\int d^3x\ e^{ipx}\overline{v}_s(0)\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}\exp\left[-\frac{1}{2}p\begin{pmatrix} \sigma^z & 0 \\ 0 & -\sigma^z \end{pmatrix}\right]\gamma^0\Psi(x)|0\rangle$$

Writing this differently:

$$J_z b_s^\dagger(p\hat{z})|0\rangle = \frac{1}{2}\int d^3x\ e^{ipx}\overline{\Psi}(x)\gamma^0\exp\left(i\eta p K^z\right)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} u_s(0)|0\rangle$$

$$J_z d_s^\dagger(p\hat{z})|0\rangle = -\frac{1}{2}\int d^3x\ e^{ipx}\overline{v}_s(0)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\exp\left(i\eta p K^z\right)\gamma^0\Psi(x)|0\rangle$$

Looking at equation 38.6 and doing the matrix multiplication, we see that the matrix acting on the spinor gives s times the spinor. Thus:

$$J_z b_s^\dagger(p\hat{z})|0\rangle = \frac{s}{2}\int d^3x\ e^{ipx}\overline{\Psi}(x)\gamma^0 \exp\left(i\eta p K^z\right) u_s(0)|0\rangle$$

$$J_z d_s^\dagger(p\hat{z})|0\rangle = -\frac{s}{2}\int d^3x\ e^{ipx}\overline{v}_s(0) \exp\left(i\eta p K^z\right)\gamma^0\Psi(x)|0\rangle$$

Using 39.10 and 39.12, we have:

$$J_z b_s^\dagger(p\hat{z})|0\rangle = \frac{1}{2}s\ b_s^\dagger(p\hat{z})|0\rangle$$

$$J_z d_s^\dagger(p\hat{z})|0\rangle = -\frac{1}{2}s\ d_s^\dagger(p\hat{z})|0\rangle$$

as expected.

Srednicki 39.3.

(a) Show that

$$\boldsymbol{U(\Lambda)^{-1}b_s(\vec{p})U(\Lambda) = \sum_{s'} R_{ss'}(\Lambda,\vec{p})b_{s'}(\Lambda^{-1}\vec{p})}$$

$$\boldsymbol{U(\Lambda)^{-1}d_s(\vec{p})U(\Lambda) = \sum_{s'} \widetilde{R}_{ss'}(\Lambda,\vec{p})d_{s'}(\Lambda^{-1}\vec{p})}$$

and find formulae for $\boldsymbol{R_{ss'}(\Lambda,\vec{p})}$ and $\boldsymbol{\tilde{R}_{ss'}(\Lambda,\vec{p})}$ that involve matrix elements of $\boldsymbol{D(\Lambda)}$ between appropriate $\boldsymbol{u}$ and $\boldsymbol{v}$ spinors.

Much of this was already done for scalars in problem 3.3. Let's start from the beginning, however, since problem 3.3 was very difficult. We'll also break this down into manageable pieces as before.

(i) Evaluate the Fourier transform of the fermionic field operator

Using the four-dimensional Fourier Transform:

$$U(\Lambda)^{-1}\Psi(k)U(\Lambda) = U(\Lambda)^{-1}\int d^4x e^{-ipx}\Psi(x)U(\Lambda)$$

d^4x is invariant under Lorentz Transformations, so are the constants, and so is the product of two four-vectors. Hence:

$$U(\Lambda)^{-1}\Psi(k)U(\Lambda) = \int d^4x e^{-ipx}U(\Lambda)^{-1}\Psi(x)U(\Lambda)$$

Now we can use equation 36.54:

$$U(\Lambda)^{-1}\Psi(k)U(\Lambda) = \int d^4x e^{-ipx}D(\Lambda)\Psi(\Lambda^{-1}x)$$

Now define $y = \Lambda^{-1}x$.

$$U(\Lambda)^{-1}\Psi(k)U(\Lambda) = \int d^4x e^{-ik\Lambda y} D(\Lambda)\Psi(y)$$

In the exponent, we now have $k\Lambda y = k^\mu \Lambda_\mu^{\ \nu} y_\nu$. Let's now use equation 2.5: this gives $k^\mu (\Lambda^{-1})^\nu_{\ \mu} y_\nu = y\Lambda^{-1}k$. This gives:

$$U(\Lambda)^{-1}\Psi(k)U(\Lambda) = \int d^4x e^{-iy\Lambda^{-1}k} D(\Lambda)\Psi(y)$$

Since the determinant of Λ is required to be one by propriety, we have:

$$U(\Lambda)^{-1}\Psi(k)U(\Lambda) = \int d^4y e^{-iy\Lambda^{-1}k} D(\Lambda)\Psi(y)$$

Fourier transforming the right-hand side, we obtain:

$$U(\Lambda)^{-1}\Psi(k)U(\Lambda) = D(\Lambda)\Psi(\Lambda^{-1}k)$$

(ii) Verify that $\boldsymbol{\Psi(k) = \sum_s 2\pi\delta(p^2+m^2)\left[\theta(p^0)u_s(\vec{p})b_s(\vec{p}) + \theta(-p^0)v_s(\vec{p})d_s^\dagger(\vec{p})\right]}$

First, how do we know that this is what we need to prove? This is a somewhat organic process: we look at the solution to problem 3.3 to see the general form that would be helpful moving forward, and we look at equation 37.30 to see the form to which our expression must simplify.

We begin the verification with the Fourier Transform:

$$\Psi(x) = \int \frac{d^4k}{(2\pi)^4} e^{ipx}\Psi(k)$$

Then we insert the claim:

$$\Psi(x) = \int \frac{d^4p}{(2\pi)^4} e^{ipx} \sum_s 2\pi\delta(p^2+m^2)\left[\theta(p^0)u_s(\vec{p})b_s(\vec{p}) + \theta(-p^0)v_s(\vec{p})d_s^\dagger(\vec{p})\right]$$

which is:

$$\Psi(x) = \sum_s \int \frac{d^4p}{(2\pi)^3} e^{ipx}\delta(p^2+m^2)\left[\theta(p^0)u_s(\vec{p})b_s(\vec{p}) + \theta(-p^0)v_s(\vec{p})d_s^\dagger(\vec{p})\right]$$

Now recall that we can break down delta functions according to $\delta(g(x)) = \sum_{x_0} \frac{\delta(x-x_0)}{|g'(x_0)|}$. Ignoring the parts that are not allowed due to the theta function, this becomes:

$$\Psi(x) = \sum_s \int \frac{d^4k}{(2\pi)^3} e^{ipx}\left[\frac{\delta(\omega - \sqrt{p^2+m^2})}{2\omega}\theta(p^0)u_s(\vec{p})b_s(\vec{p}) + \frac{\delta(\omega + \sqrt{p^2+m^2})}{2\omega}\theta(-p^0)v_s(\vec{p})d_s^\dagger(\vec{p})\right]$$

which is:

$$\Psi(x) = \sum_s \int d\omega \widetilde{dp} e^{ipx}\left[\delta(\omega - \sqrt{p^2+m^2})\theta(p^0)u_s(\vec{p})b_s(\vec{p}) + \delta(\omega + \sqrt{p^2+m^2})\theta(-p^0)v_s(\vec{p})d_s^\dagger(\vec{p})\right]$$

The theta functions are no longer necessary, since the signs are appropriately specified by the delta functions:

$$\Psi(x) = \sum_s \int d\omega \widetilde{dp} e^{ipx} \left[\delta(\omega - \sqrt{p^2+m^2}) u_s(\vec{p}) b_s(\vec{p}) + \delta(\omega + \sqrt{p^2+m^2}) v_s(\vec{p}) d_s^\dagger(\vec{p}) \right]$$

Now we could use the delta function to eliminate k^0 from the equation. Instead, let's continue to use ω, though we now must remember that ω is completely defined by p and m. Further, let's define ω to be $+\sqrt{k^2+m^2}$: this causes ω to become $-\omega$ in the second term. Thus, our delta functions give:

$$\Psi(x) = \sum_s \int \widetilde{dp} \left[e^{ipx} u_s(\vec{p}) b_s(\vec{p}) + e^{i\omega t} e^{i\vec{p}\cdot\vec{p}} v_s(\vec{p}) d_s^\dagger(\vec{p}) \right]$$

This is a bit strained notation, so let's redefine the dummy variable in the second term, $\vec{p} \to -\vec{p}$.

$$\Psi(x) = \sum_s \int \widetilde{dp} \left[e^{ipx} u_s(\vec{p}) b_s(\vec{p}) + e^{-ipx} v_s(\vec{p}) d_s^\dagger(\vec{p}) \right]$$

which is equation 37.30, verifying the claim.

(iii) Take Lorentz Transforms of the result of part (b)

$$U(\Lambda)^{-1} \Psi(k) U(\Lambda) = U(\Lambda)^{-1} \sum_s 2\pi\delta(p^2+m^2) \left[\theta(p^0) u_s(\vec{p}) b_s(\vec{p}) + \theta(-p^0) v_s(\vec{p}) d_s^\dagger(\vec{p}) \right] U(\Lambda)$$

Let's start by taking p^0 to be positive. This gives:

$$U(\Lambda)^{-1} \Psi(k) U(\Lambda) = 2\pi\delta(p^2+m^2) U(\Lambda)^{-1} \sum_s u_s(\vec{p}) b_s(\vec{p}) U(\Lambda)$$

On the left-hand side, we use the result from part (a).

$$D(\Lambda)\Psi(\Lambda^{-1}k) = 2\pi\delta(p^2+m^2) U(\Lambda)^{-1} \sum_s u_s(\vec{p}) b_s(\vec{p}) U(\Lambda)$$

Now we use the result from part (b) again on the left-hand side:

$$D(\Lambda) 2\pi\delta(p^2+m^2) \sum_s u_s(\Lambda^{-1}\vec{p}) b_s(\Lambda^{-1}\vec{p}) = 2\pi\delta(p^2+m^2) U(\Lambda)^{-1} \sum_s u_s(\vec{p}) b_s(\vec{p}) U(\Lambda)$$

where the p^2 in the delta function is the magnitude of a four-vector, and therefore invariant.

We cancel the factors of 2π and equate the coefficients of a delta-function (this is allowed; it's a Green's expansion!). This gives:

$$U(\Lambda)^{-1} \sum_s u_s(\vec{p}) b_s(\vec{p}) U(\Lambda) = \sum_s D(\Lambda) u_s(\Lambda^{-1}\vec{p}) b_s(\Lambda^{-1}\vec{p}) \qquad (39.3.1)$$

Now we multiply on the left of both sides by $\overline{u}_{s'}(\vec{p})$. Let's also reverse the direction of the equality:

$$\overline{u}_{s'}(\vec{p})U(\Lambda)^{-1}\sum_s u_s(\vec{p})b_s(\vec{p})U(\Lambda) = \overline{u}_{s'}(\vec{p})\sum_s D(\Lambda)u_s(\Lambda^{-1}\vec{p})b_s(\Lambda^{-1}\vec{p})$$

Recall that the spinors are NOT four-vectors: they are spinors which depend only on the 3-vector $\vec{k}$. In other words, they are transformed by $D(\Lambda)$, not by $U(\Lambda)$. Hence, we can simply move the spinor on the left hand side past the $U(\Lambda)^{-1}$, in order to obtain:

$$U(\Lambda)^{-1}\sum_s \overline{u}_{s'}(\vec{p})u_s(\vec{p})b_s(\vec{p})U(\Lambda) = \overline{u}_{s'}(\vec{p})\sum_s D(\Lambda)u_s(\Lambda^{-1}\vec{p})b_s(\Lambda^{-1}\vec{p})$$

Now we can use 38.17 (we'll choose s as our index on the left-hand side – to be consistent, we'll relabel on the right hand side as well):

$$U(\Lambda)^{-1}b_s(\vec{p})U(\Lambda) = \frac{1}{2m}\overline{u}_s(\vec{p})\sum_{s'} D(\Lambda)u_{s'}(\Lambda^{-1}\vec{p})b_{s'}(\Lambda^{-1}\vec{p})$$

Rewriting a bit more, we have:

$$U(\Lambda)^{-1}b_s(\vec{p})U(\Lambda) = \sum_{s'}\left[\frac{1}{2m}\overline{u}_s(\vec{p})D(\Lambda)u_{s'}(\Lambda^{-1}\vec{p})\right]b_{s'}(\Lambda^{-1}\vec{p})$$

which we write as:

$$U(\Lambda)^{-1}b_s(\vec{p})U(\Lambda) = \sum_{s'} R_{ss'}(\Lambda,\vec{p})b_{s'}(\Lambda^{-1}\vec{p})$$

$\square$

What about if we choose p^0 to be negative? Then we have the same thing, with $u \to v$ and $b \to d^\dagger$. We can therefore read off the answer:

$$U(\Lambda)^{-1}d_s^\dagger(\vec{p})U(\Lambda) = \sum_{s'}\left[\frac{1}{2m}\overline{v}_s(\vec{p})D(\Lambda)v_{s'}(\Lambda^{-1}\vec{p})\right]d_{s'}^\dagger(\Lambda^{-1}\vec{p})$$

Take the Hermitian conjugate of both sides:

$$U(\Lambda)^{-1}d_s(\vec{p})U(\Lambda) = \sum_{s'}\left[\frac{1}{2m}\overline{v}_s(\vec{p})D(\Lambda)v_{s'}(\Lambda^{-1}\vec{p})\right]^* d_{s'}(\Lambda^{-1}\vec{p})$$

which we write as:

$$U(\Lambda)^{-1}d_s(\vec{p})U(\Lambda) = \sum_{s'} \tilde{R}_{ss'}(\Lambda,\vec{p})d_{s'}(\Lambda^{-1}\vec{p})$$

$\square$

(b) Show that $\tilde{R}_{ss'}(\Lambda, \vec{p}) = R_{ss'}(\Lambda, \vec{p})$

We have to show that:

$$R_{ss'}(\Lambda, \vec{p}) = \tilde{R}_{ss'}(\Lambda, \vec{p})$$

which is:

$$\frac{1}{2m}\overline{u}_s(\vec{p})D(\Lambda)u_{s'}(\Lambda^{-1}\vec{p}) = \frac{1}{2m}\left[\overline{v}_s(\vec{p})D(\Lambda)v_{s'}(\Lambda^{-1}\vec{p})\right]^*$$
$$\overline{u}_s(\vec{p})D(\Lambda)u_{s'}(\Lambda^{-1}\vec{p}) = \left[\overline{v}_s(\vec{p})D(\Lambda)v_{s'}(\Lambda^{-1}\vec{p})\right]^*$$
$$\overline{u}_s(\vec{p})D(\Lambda)u_{s'}(\Lambda^{-1}\vec{p}) = v_{s'}(\Lambda^{-1}\vec{p})^*D(\Lambda)^*\overline{v}_s(\vec{p})^*$$

Recall that the complex conjugate of a spinor is the barred spinor. Thus:

$$\overline{u}_s(\vec{p})D(\Lambda)u_{s'}(\Lambda^{-1}\vec{p}) = \overline{v}_{s'}(\Lambda^{-1}\vec{p})D(\Lambda)^*v_s(\vec{p})$$

From the form of $D(\Lambda)$, it is clear that:

$$\overline{u}_s(\vec{p})D(\Lambda)u_{s'}(\Lambda^{-1}\vec{p}) = \overline{v}_{s'}(\Lambda^{-1}\vec{p})D(\Lambda^{-1})v_s(\vec{p})$$

This is just one number, so we can take the transpose without any difficulty:

$$\overline{u}_s(\vec{p})D(\Lambda)u_{s'}(\Lambda^{-1}\vec{p}) = \left[\overline{v}_{s'}(\Lambda^{-1}\vec{p})D(\Lambda^{-1})v_s(\vec{p})\right]^T$$

which is:

$$\overline{u}_s(\vec{p})D(\Lambda)u_{s'}(\Lambda^{-1}\vec{p}) = v_s(\vec{p})^TD(\Lambda^{-1})^T\overline{v}_{s'}(\Lambda^{-1}\vec{p})^T$$

Now we can insert some identities:

$$\overline{u}_s(\vec{p})D(\Lambda)u_{s'}(\Lambda^{-1}\vec{p}) = v_s(\vec{p})^T\mathcal{C}\mathcal{C}^{-1}D(\Lambda^{-1})^T\mathcal{C}\mathcal{C}^{-1}\overline{v}_{s'}(\Lambda^{-1}\vec{p})^T$$

Using 39.28 and its conjugate:

$$\overline{u}_s(\vec{p})D(\Lambda)u_{s'}(\Lambda^{-1}\vec{p}) = \bar{u}_s(\vec{p})\mathcal{C}^{-1}D(\Lambda^{-1})^T\mathcal{C}u_{s'}(\Lambda^{-1}\vec{p})$$

$D(\Lambda)$ consists of an exponential (an infinite series) of constants and the product of two gamma matrices. The constants are invariant; the gamma matrices transform according to equation 38.36. Since there are two gamma matrices, the negative sign cancels; the transposes also cancel with the transpose in the equation. Thus:

$$\overline{u}_s(\vec{p})D(\Lambda)u_{s'}(\Lambda^{-1}\vec{p}) = \bar{u}_s(\vec{p})D(\Lambda)u_{s'}(\Lambda^{-1}\vec{p})$$

as expected.

(c) Show that

$$\boldsymbol{U(\Lambda)|p, s, q\rangle = \sum_{s'} R^*_{ss'}(\Lambda^{-1}, \vec{p})|\Lambda p, s', q\rangle}$$

where

$$\boldsymbol{|p, s, +\rangle = b^\dagger_s(\vec{p})|0\rangle}$$
$$\boldsymbol{|p, s, -\rangle = d^\dagger_s(\vec{p})|0\rangle}$$

are single one-particle states.

We have:

$$U(\Lambda)|p,s,+\rangle = U(\Lambda)b_s^\dagger(\vec{p})|0\rangle$$

We insert the identity:

$$U(\Lambda)|p,s,+\rangle = U(\Lambda)b_s^\dagger(\vec{p})U(\Lambda)^{-1}U(\Lambda)|0\rangle$$

The $U(\Lambda)$ operator acting on the vacuum won't do much:

$$U(\Lambda)|p,s,+\rangle = U(\Lambda)b_s^\dagger(\vec{p})U(\Lambda)^{-1}|0\rangle \tag{39.3.2}$$

Equation 39.39 gives:

$$U(\Lambda)^{-1}b_s(\vec{p})U(\Lambda) = \sum_{s'} R_{ss'}(\Lambda,\vec{p})b_{s'}(\Lambda^{-1}\vec{p})$$

Taking the Hermitian conjugate gives:

$$U(\Lambda)^{-1}b_s^\dagger(\vec{p})U(\Lambda) = \sum_{s'} R^*_{ss'}(\Lambda,\vec{p})b^\dagger_{s'}(\Lambda^{-1}\vec{p})$$

Now let's reverse the direction of the Lorentz Transformation. This will give:

$$U(\Lambda)b_s^\dagger(\vec{p})U(\Lambda)^{-1} = \sum_{s'} R^*_{ss'}(\Lambda^{-1},\vec{p})b^\dagger_{s'}(\Lambda\vec{p})$$

Using this in equation (39.3.2) gives:

$$U(\Lambda)|p,s,+\rangle = \sum_{s'} R^*_{ss'}(\Lambda^{-1},\vec{p})b^\dagger_{s'}(\Lambda\vec{p})|0\rangle$$

This gives:

$$U(\Lambda)|p,s,+\rangle = \sum_{s'} R^*_{ss'}(\Lambda^{-1},\vec{p})|\Lambda p,s,+\rangle$$

Absolutely nothing will change if we take $+\rightarrow -$. Combining both cases, we have:

$$U(\Lambda)|p,s,q\rangle = \sum_{s'} R^*_{ss'}(\Lambda^{-1},\vec{p})|\Lambda p,s,q\rangle$$

as expected.

Note: this is one of the hardest problems so far. I spent a lot of time trying to think through the subtleties and provide a detailed explanation. If anything is unclear, please feel free to e-mail.

Srednicki 39.4. The Spin Statistics Theorem for spin-one-half particles. We will follow the proof for spin-zero particles in section 4. We start with $\boldsymbol{b_s(\vec{p})}$ and

$b_s^\dagger(\vec{p})$ as the fundamental objects; we take them to have either commutation (-) or anticommutation (+) relations of the form:

$$[b_s(\vec{p}), b_{s'}(\vec{p'})]_\mp = 0$$

$$[b_s^\dagger(\vec{p}), b_{s'}^\dagger(\vec{p'})]_\mp = 0$$

$$[b_s(\vec{p}), b_{s'}^\dagger(\vec{p'})]_\mp = (2\pi)^3 2\omega\delta^3(\vec{p} - \vec{p'})\delta_{ss'}$$

Define

$$\Psi^+(x) = \sum_{s=\pm} \int \widetilde{dp}\ b_s(\vec{p}) u_s(\vec{p}) e^{ipx}$$

$$\Psi^-(x) = \sum_{s=\pm} \int \widetilde{dp}\ b_s^\dagger(\vec{p}) v_s(\vec{p}) e^{-ipx}$$

(a) Show that $U(\Lambda)^{-1}\Psi^\pm(x)U(\Lambda) = D(\Lambda)\Psi^\pm(\Lambda^{-1}x)$

We have:

$$U(\Lambda)^{-1}\Psi^+(x)U(\Lambda) = U(\Lambda)^{-1}\sum_{s=\pm}\int \widetilde{dp}\ b_s(\vec{p})u_s(\vec{p})e^{ipx}U(\Lambda)$$

The exponential, as the invariant product of two four-vectors, an the Lorentz-Invariant phase space, are invariant as their names suggest. Thus,

$$U(\Lambda)^{-1}\Psi^+(x)U(\Lambda) = \sum_{s=\pm}\int \widetilde{dp}\ U(\Lambda)^{-1}b_s(\vec{p})u_s(\vec{p})U(\Lambda)e^{ipx}$$

Unfortunately, we need to use equation (39.3.1). If you did not do problem 39.3(a), there's nothing for it now except to start at the beginning until you achieve equation (39.3.1). Since there is nothing that should be added or subtracted to my derivation above, I won't redo the proof, I'll just use the equation here.

This gives:

$$U(\Lambda)^{-1}\Psi^+(x)U(\Lambda) = D(\Lambda)\sum_{s=\pm}\int \widetilde{dp}\ b_s(\Lambda^{-1}\vec{p})u_s(\Lambda^{-1}\vec{p})e^{ipx}$$

Next we'll change $\vec{p} \to \Lambda\vec{p}$; this will not change the differential because it's Lorentz-Invariant. Thus:

$$U(\Lambda)^{-1}\Psi^+(x)U(\Lambda) = D(\Lambda)\sum_{s=\pm}\int \widetilde{dp}\ b_s(\vec{p})u_s(\vec{p})e^{i\Lambda px}$$

Next we'll play with the indices: $\Lambda^\mu{}_\nu p^\nu x_\mu = (\Lambda^{-1})_\nu{}^\mu p^\nu x_\mu = p\Lambda^{-1}x$, where the first equality follows by equation 2.5. Then:

$$U(\Lambda)^{-1}\Psi^+(x)U(\Lambda) = D(\Lambda)\sum_{s=\pm}\int \widetilde{dp}\ b_s(\vec{p})u_s(\vec{p})e^{ip\Lambda^{-1}x}$$

which is:

$$U(\Lambda)^{-1}\Psi^+(x)U(\Lambda) = D(\Lambda)\Psi^+(\Lambda^{-1}x)$$

Nothing in our argument is changed if we change the b to $b^\dagger$ or the u to a v. Are we allowed to make this substitution in equation (39.3.1)? Yes: examine the proof of (39.3.1). At the beginning of part (iii) we took p^0 to be positive; now we take it to be negative. This allows us to swap $u \leftrightarrow v$, which is good, and $b \leftrightarrow d^\dagger$. To get from $d^\dagger$ to $b^\dagger$, we simply consider the case of a Majorana fermion, proving the identity (remember the identity is just a relation between a bunch of spinors; considering the case of a Majorana fermion does not mean that the identity only holds in the case of a Majorana fermion). Thus:

$$U(\Lambda)^{-1}\Psi^{\pm}(x)U(\Lambda) = D(\Lambda)\Psi^{\pm}(\Lambda^{-1}x)$$

as expected.

(b) Show that $[\Psi^+(x)]^\dagger = [\Psi^-(x)]^T\mathcal{C}\beta$. Thus an hermitian interaction term in the Lagrangian must involve both Ψ^+ and Ψ^-.

We have:

$$[\Psi^+(x)]^\dagger = \sum_s \int \widetilde{dp}\; e^{-ipx} u_s(\vec{p})^\dagger b_s(\vec{p})^\dagger$$

This is:

$$[\Psi^+(x)]^\dagger = \sum_s \int \widetilde{dp}\; e^{-ipx} [u_s(\vec{p})]^{*T} b_s(\vec{p})^\dagger$$

Using equation 38.38:

$$[\Psi^+(x)]^\dagger = \sum_s \int \widetilde{dp}\; e^{-ipx} [\mathcal{C}\beta v_s(\vec{p})]^T b_s(\vec{p})^\dagger$$

which is:

$$[\Psi^+(x)]^\dagger = \sum_s \int \widetilde{dp}\; e^{-ipx} v_s(\vec{p})^T \beta^T \mathcal{C}^T b_s(\vec{p})^\dagger$$

β is symmetric. $\mathcal{C}^T = -\mathcal{C}$, but we kill off the negative sign by anticommuting with the β (equations 38.34 and 38.35). Thus:

$$[\Psi^+(x)]^\dagger = \sum_s \int \widetilde{dp}\; e^{-ipx} v_s(\vec{p})^T \mathcal{C}\beta b_s(\vec{p})^\dagger$$

Remember the operator will commute with all matrices. Thus:

$$[\Psi^+(x)]^\dagger = \sum_s \int \widetilde{dp}\; e^{-ipx} v_s(\vec{p})^T b_s(\vec{p})^\dagger \mathcal{C}\beta$$

which is:

$$[\Psi^+(x)]^\dagger = [\Psi^-(x)]^T\mathcal{C}\beta$$

as expected.

(c) Show that $[\Psi^+_\alpha(x), \Psi^-_\beta(y)]_\mp \neq 0$ for $(x-y)^2 > 0$.

We have:

$$[\Psi_s^+(x), \Psi_s'^-(y)]_\mp = \sum_s \sum_{s'} \int \widetilde{dp}\ \widetilde{dp'}\ e^{i(px-p'y)} [u_s(\vec{p})]_\alpha [v_s'(\vec{p'})]_\beta [b_s(\vec{p}), b_s'^\dagger(\vec{p'})]_\mp$$

Using equation 39.43:

$$[\Psi_\alpha^+(x), \Psi_\beta^-(y)]_\mp = \sum_s \int \widetilde{dp}\ e^{ip(x-y)} [u_s(\vec{p})]_\alpha [v_s(\vec{p})]_\beta$$

At the risk of stating the obvious, let me comment that α and β do not represent the spin indices, but rather the component of the field spinor. Now we recall equtaion 38.37 and 38.38:

$$v_s(\vec{p}) = \mathcal{C}\overline{u}_s(\vec{p})^T = \left(\overline{u}_s(\vec{p})\mathcal{C}^T\right)^T = -\left(\overline{u}_s(\vec{p})\mathcal{C}\right)^T$$
$$\Rightarrow v_s(\vec{p})_\beta = -\left[\overline{u}_s(\vec{p})\mathcal{C}\right]_\beta$$

Inserting this into our expression:

$$[\Psi_\alpha^+(x), \Psi_\beta^-(y)]_\mp = -\sum_s \int \widetilde{dp}\ e^{ip(x-y)} \left[u_s(\vec{p})\overline{u}_s(\vec{p})\mathcal{C}\right]_{\alpha\beta}$$

Now we use equation 38.23:

$$[\Psi_\alpha^+(x), \Psi_\beta^-(y)]_\mp = -\int \widetilde{dp}\ e^{ip(x-y)} \left[(\not{p} - m)\mathcal{C}\right]_{\alpha\beta} \tag{39.4.1}$$

Now recall that the momentum operator has nothing to do with the momentum integral; it is simply $p = -i\partial_x$. Thus:

$$[\Psi_\alpha^+(x), \Psi_\beta^-(y)]_\mp = -\left[(i\not{\partial}_x + m)\mathcal{C}\right]_{\alpha\beta} \int \widetilde{dp}\ e^{ip(x-y)}$$

Now use equation 4.12:

$$[\Psi_\alpha^+(x), \Psi_\beta^-(y)]_\mp = -\frac{m}{4\pi^2 r} \left[(i\not{\partial}_x + m)\mathcal{C}\right]_{\alpha\beta} K_1(mr)$$

As in chapter 4, we argue that this is nonzero because $K_1(mr)$ is never zero when $r^2 = (x-y)^2 > 0$ – even if the matrix element in the prefactor is zero.

(d) Show that $[\Psi_\alpha^+, \Psi_\beta^-(y)] = -[\Psi_\beta^+(y), \Psi_\alpha^-(x)]_\pm$ for $(x-y)^2 > 0$.

We go back to equation (39.4.1):

$$[\Psi_\alpha^+(x), \Psi_\beta^-(y)]_\mp = -\int \widetilde{dp}\ e^{ip(x-y)} \left[(\not{p} - m)\mathcal{C}\right]_{\alpha\beta}$$

We write this as:

$$[\Psi_\alpha^+(x), \Psi_\beta^-(y)]_\mp = -\int \widetilde{dp}\ e^{ip(x-y)} \left[(\not{p}\mathcal{C})^T - m\mathcal{C}^T\right]_{\beta\alpha}$$

$$[\Psi^+_\alpha(x), \Psi^-_\beta(y)]_\mp = -\int \widetilde{dp}\ e^{ip(x-y)} \left[(\mathcal{C}^T(\not{p})^T - m\mathcal{C}^T\right]_{\beta\alpha}$$

Using equation 38.34:

$$[\Psi^+_\alpha(x), \Psi^-_\beta(y)]_\mp = -\int \widetilde{dp}\ e^{ip(x-y)} \left[(-\mathcal{C}p_\mu(\gamma^\mu)^T + m\mathcal{C}\right]_{\beta\alpha}$$

$$[\Psi^+_\alpha(x), \Psi^-_\beta(y)]_\mp = -\int \widetilde{dp}\ e^{ip(x-y)} \left[(-p_\mu\mathcal{C}(\gamma^\mu)^T + m\mathcal{C}\right]_{\beta\alpha}$$

Using equation 38.36:

$$[\Psi^+_\alpha(x), \Psi^-_\beta(y)]_\mp = -\int \widetilde{dp}\ e^{ip(x-y)} \left[(p_\mu(\gamma^\mu)\mathcal{C} + m\mathcal{C}\right]_{\beta\alpha}$$

which is:

$$[\Psi^+_\alpha(x), \Psi^-_\beta(y)]_\mp = -\int \widetilde{dp}\ e^{ip(x-y)} \left[(\not{p} + m)\mathcal{C}\right]_{\beta\alpha}$$

Now we use the fact that $(x-y)^2 > 0$: this is a spacelike separation between the two "events," meaning that we can find a frame in which the two events are simultaneous. Thus:

$$[\Psi^+_\alpha(x), \Psi^-_\beta(y)]_\mp = -\int \widetilde{dp}\ e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \left[(\not{p} + m)\mathcal{C}\right]_{\beta\alpha}$$

which is:

$$[\Psi^+_\alpha(x), \Psi^-_\beta(y)]_\mp = -\int \widetilde{dp}\ e^{i(-\vec{p})\cdot(\vec{y}-\vec{x})} \left[(\not{p} + m)\mathcal{C}\right]_{\beta\alpha}$$

Now we switch the integration variable from $\vec{p}$ to $-\vec{p}$; this does not change the integration measure because we're still integrating over the same space. Thus:

$$[\Psi^+_\alpha(x), \Psi^-_\beta(y)]_\mp = -\int \widetilde{dp}\ e^{i(\vec{p})\cdot(\vec{y}-\vec{x})} \left[(-\not{p} + m)\mathcal{C}\right]_{\beta\alpha}$$

Now we distribute the negative sign and reinsert the temporal part of the index:

$$[\Psi^+_\alpha(x), \Psi^-_\beta(y)]_\mp = \int \widetilde{dp}\ e^{ip(y-x)} \left[(\not{p} - m)\mathcal{C}\right]_{\beta\alpha}$$

The right-hand side of this is, by equation (39.4.1):

$$[\Psi^+_\alpha(x), \Psi^-_\beta(y)]_\mp = [\Psi^+_\beta(y), \Psi^-_\alpha(x)]_\mp$$

as expected.

(e) Consider $\Psi(x) = \Psi^+(x) + \lambda\Psi^-(x)$, where λ is an arbitrary complex number, and evaluate both $[\Psi_\alpha(x), \Psi_\beta y)]_\mp$ and $[\Psi_\alpha(x), \overline{\Psi}_\beta(y)]_\mp$ for $(x-y)^2 > 0$. Show these can both vanish if and only if $|\lambda| = 1$ and we use anticommutators.

$$[\Psi_\alpha(x), \Psi_\beta(y)]_\mp = \left[\Psi^+_\alpha(x) + \lambda\Psi^-_\alpha(x), \Psi^+_\beta(y) + \lambda\Psi^-_\beta(y)\right]_\mp$$

Using 39.43, we can simplify this as:

$$[\Psi_\alpha(x), \Psi_\beta(y)]_\mp = \lambda[\Psi^+_\alpha(x), \Psi^-_\beta(y)]_\mp + \lambda[\Psi^-_\alpha(x), \Psi^+_\beta(y)]_\mp$$

We use the results of part (d):

$$[\Psi_\alpha(x), \Psi_\beta(y)]_\mp = \lambda[\Psi^+_\alpha(x), \Psi^-_\beta(y)]_\mp - \lambda[\Psi^-_\beta(y), \Psi^+_\alpha(x)]_\mp$$

Of course, $[A, B] = AB - BA = -(BA - AB) = -[B, A]$ and $\{A, B\} = AB + BA = BA + AB = \{B, A\}$. This gives:

$$[\Psi_\alpha(x), \Psi_\beta(y)]_\mp = \lambda[\Psi^+_\alpha(x), \Psi^-_\beta(y)]_\mp \pm \lambda[\Psi^+_\alpha(x), \Psi^-_\beta(y)]_\mp$$

$$[\Psi_\alpha(x), \Psi_\beta(y)]_\mp = \lambda(1 \pm 1)[\Psi^+_\alpha(x), \Psi^-_\beta(y)]_\mp$$

Using the result of part (c), there is no way that these will commute. They will anticommute if and only if we accept canonical anticommutation relations.

For the second commutator in question, we have:

$$[\Psi_\alpha(x), \overline{\Psi}_\beta(x)]_\mp = [\Psi_\alpha(x), \Psi^\dagger_\beta(y)\beta]_\mp$$

$$[\Psi_\alpha(x), \overline{\Psi}_\beta(x)]_\mp = [\Psi^+_\alpha(x) + \lambda\Psi^-_\alpha(x), \Psi^+_\beta(y)^\dagger\beta + \lambda^*\Psi^-_\beta(y)^\dagger\beta]_\mp$$

Multiplying these out, the first and last terms have only bs; the second and third have only $b^\dagger$s. Thus, the only pairings that will not vanish are the first and third, and the second and fourth:

$$[\Psi_\alpha(x), \overline{\Psi}_\beta(x)]_\mp = [\Psi^+_\alpha(x), \Psi^+_\beta(y)^\dagger\beta] + |\lambda^2|[\Psi^-_\alpha(x), \Psi^-_\beta(y)^\dagger\beta]$$

Using equation 39.44:

$$[\Psi_\alpha(x), \overline{\Psi}_\beta(x)]_\mp = \sum_s \sum_{s'} \int \widetilde{dp}\ \widetilde{dp'}\ u_s(\vec{p})_\alpha \overline{u}_{s'}(\vec{p})_\beta e^{i(px-p'y)}[b_s(\vec{p}), b^\dagger_s(\vec{p})]_\mp$$

$$+|\lambda|^2 \sum_s \sum_{s'} \int \widetilde{dp}\ \widetilde{dp'}\ v_s(\vec{p})_\alpha \overline{v}_{s'}(\vec{p})_\beta e^{-i(px-p'y)}[b^\dagger_s(\vec{p}), b_s(\vec{p})]_\mp$$

As in part (d), we note that $[A, B]_\mp = \mp[B, A]_\mp$. Thus:

$$[\Psi_\alpha(x), \overline{\Psi}_\beta(x)]_\mp = \sum_s \sum_{s'} \int \widetilde{dp}\ \widetilde{dp'}\ u_s(\vec{p})_\alpha \overline{u}_{s'}(\vec{p})_\beta e^{i(px-p'y)}[b_s(\vec{p}), b^\dagger_s(\vec{p})]_\mp$$

$$\mp|\lambda|^2 \sum_s \sum_{s'} \int \widetilde{dp}\ \widetilde{dp'}\ v_s(\vec{p})_\alpha \overline{v}_{s'}(\vec{p})_\beta e^{-i(px-p'y)}[b_s(\vec{p}), b^\dagger_s(\vec{p})]_\mp$$

Using the commutator and equation 39.43:

$$[\Psi_\alpha(x), \overline{\Psi}_\beta(x)]_\mp = \sum_s \int \widetilde{dp}\ u_s(\vec{p})_\alpha \overline{u}_{s'}(\vec{p})_\beta e^{ip(x-y)} \mp |\lambda|^2 \sum_s \int \widetilde{dp}\ v_s(\vec{p})_\alpha \overline{v}_{s'}(\vec{p})_\beta e^{-ip(x-y)}$$

Now we use 38.23:

$$[\Psi_\alpha(x), \overline{\Psi}_\beta(x)]_\mp = \int \widetilde{dp}\ (-\not{p} + m)_{\alpha\beta} e^{ip(x-y)} \mp |\lambda|^2 \int \widetilde{dp}\ (-\not{p} - m)_{\alpha\beta} e^{-ip(x-y)}$$

In the second term, we take the $\vec{k} \to -\vec{k}$; as usual this does not affect the integration measure. Thus:

$$[\Psi_\alpha(x), \overline{\Psi}_\beta(x)]_\mp = \int \widetilde{dp}\ (-\not{p} + m)_{\alpha\beta} e^{ip(x-y)} \mp |\lambda|^2 \int \widetilde{dp}\ (\not{p} - m)_{\alpha\beta} e^{ip(x-y)}$$

which is:

$$[\Psi_\alpha(x), \overline{\Psi}_\beta(x)]_\mp = (1 \pm |\lambda|^2) \int \widetilde{dp}\ (-\not{p} + m)_{\alpha\beta} e^{ip(x-y)}$$

Now $/p = p^\mu \gamma_\mu$: as the contraction of two four-vectors, the product is invariant as we integrate over the Lorentz-Invariant phase space. Thus,

$$[\Psi_\alpha(x), \overline{\Psi}_\beta(x)]_\mp = (1 \pm |\lambda|^2)(-\not{p} + m)_{\alpha\beta} \int \widetilde{dp}\ e^{ip(x-y)}$$

$p_\mu = -i\partial_\mu$. Further, we use equation 4.12:

$$[\Psi_\alpha(x), \overline{\Psi}_\beta(x)]_\mp = (1 \pm |\lambda|^2)(i\not{\partial} + m)_{\alpha\beta} C(r)$$

which will vanish only if and only if we choose anticommutators and require $|\lambda| = 1$.