Srednicki Chapter 38

QFT Problems & Solutions

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Srednicki 38.1. Use equation 38.12 to compute $u_s(p)$ and $v_s(p)$ explicitly. Hint: Show that the matrix $2i\hat{p}\cdot K$ has eigenvalues ± 1 and that, for any matrix A with eigenvalues ± 1, $e^{cA} = \cosh c + (\sinh c)A$, where c is an arbitrary complex number.

We have:

$$2iK^j = -\gamma^i\gamma^0$$

$$2iK^j = -\begin{pmatrix} 0 & \gamma^j \\ -\gamma^j & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} -\sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix}$$

At the risk of stating the obvious, note that this represents three matrices, each one indicated by j. Each of these matrices is has eigenvalues ± 1. Any linear combination (of unit magnitude) of these matrices will similarly have eigenvalues ± 1. Thus, $2i\hat{p}\cdot K$ will be a single matrix with eigenvalues ± 1.

Next, we simply expand:

$$\exp(cA) = \sum_{\text{n even}} \frac{(cA)^n}{n!} + \sum_{\text{n odd}} \frac{(cA)^n}{n!}$$

Working in a basis where A is diagonal, we have $A^2 = 1$ (recall that we defined A to have eigenvalues ± 1). Then:

$$\exp(cA) = \sum_{\text{n even}} \frac{(c)^n}{n!} + \sum_{\text{n odd}} \frac{(c)^n}{n!}A$$

This gives:

$$\exp(cA) = \cosh c + (\sinh c)A$$

Combining these results, we have:

$$\exp(i\eta\hat{p}\cdot K) = \cosh\left(\frac{\eta}{2}\right) + \sinh\left(\frac{\eta}{2}\right)\begin{pmatrix} -\hat{p}\cdot\vec{\sigma} & 0 \\ 0 & \hat{p}\cdot\vec{\sigma} \end{pmatrix}$$

We can write this as one matrix:

$$\exp(i\eta\hat{p}\cdot K) = \begin{pmatrix} \cosh(\eta/2) - \sinh(\eta/2)\hat{p}\cdot\vec{\sigma} & 0 \\ 0 & \cosh(\eta/2) + \sinh(\eta/2)\hat{p}\cdot\vec{\sigma} \end{pmatrix} \qquad (38.1.1)$$

To simplify this, we need to do something with $\hat{p}\cdot\vec{\sigma}$. Let's expand:

$$
\begin{aligned}
\hat{p}\cdot\vec{\sigma} &= p_x\sigma_1 + p_y\sigma_2 + p_z\sigma_3 \\
&= p\sin\theta\cos\phi\ \sigma_1 + p\sin\theta\sin\phi\ \sigma_2 + p\cos\theta\ \sigma_3 \\
&= p\sin\theta\cos\phi\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + p\sin\theta\sin\phi\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + p\cos\theta\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= p\sin\theta\begin{pmatrix} 0 & \cos\phi - i\sin\phi \\ \cos\phi + i\sin\phi & 0 \end{pmatrix} + p\cos\theta\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= p\sin\theta\begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix} + p\cos\theta\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= p\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}
\end{aligned}
$$

Combining this with equation (38.1.1), we have:

$$
\exp(i\eta\hat{p}\cdot K) = \begin{pmatrix} \cosh(\eta/2) - p\sinh(\eta/2)\cos\theta & -p\sinh(\eta/2)\sin\theta e^{-i\phi} & 0 & 0 \\ -p\sinh(\eta/2)\sin\theta e^{i\phi} & \cosh(\eta/2) + p\sinh(\eta/2)\cos\theta & 0 & 0 \\ 0 & 0 & cosh(\eta/2) + p\sinh(\eta/2)\cos\theta & p\sinh(\eta/2)\sin\theta e^{-i\phi} \\ 0 & 0 & p\sinh(\eta/2)\sin\theta e^{i\phi} & \cosh(\eta/2) - p\sinh(\eta/2)\cos\theta \end{pmatrix}
$$

Now we just have to multiply some matrices, according to equation 38.12. The result is:

$$
\boxed{u_+(\vec{p}) = \sqrt{m}\begin{pmatrix} \cosh(\eta/2) - p\sinh(\eta/2)\cos\theta \\ -p\sinh(\eta/2)\sin\theta e^{i\phi} \\ \cosh(\eta/2) + p\sinh(\eta/2)\cosh\theta \\ p\sinh(\eta/2)\sin\theta e^{i\phi} \end{pmatrix}}
$$

$$
\boxed{u_-(\vec{p}) = \sqrt{m}\begin{pmatrix} -p\sinh(\eta/2)\sin\theta e^{-i\phi} \\ \cosh(\eta/2) + p\sinh(\eta/2)\cos\theta \\ p\sinh(\eta/2)\sin\theta e^{-i\phi} \\ \cosh(\eta/2) - p\sinh(\eta/2)\cos\theta \end{pmatrix}}
$$

$$
\boxed{v_+(\vec{p}) = \sqrt{m}\begin{pmatrix} -p\sinh(\eta/2)\sin\theta e^{-i\phi} \\ \cosh(\eta/2) + p\sinh(\eta/2)\cos\theta \\ -p\sinh(\eta/2)\sin\theta e^{-i\phi} \\ -\cosh(\eta/2) + p\sinh(\eta/2)\cos\theta \end{pmatrix}}
$$

$$
\boxed{v_-(\vec{p}) = \sqrt{m}\begin{pmatrix} -\cosh(\eta/2) + p\sinh(\eta/2)\cos\theta \\ p\sinh(\eta/2)\sin\theta e^{i\phi} \\ \cosh(\eta/2) + p\sinh(\eta/2)\cosh\theta \\ p\sinh(\eta/2)\sin\theta e^{i\phi} \end{pmatrix}}
$$

Srednicki 38.2. Verify equation 38.15.

We have:

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

So:

$$\gamma^{0\dagger} = \gamma^0$$

$$\gamma^{i\dagger} = -\gamma^i$$

since the Pauli matrices are Hermitian.

Then we have:

$$\overline{\gamma^0} = \beta\gamma^0\beta$$

$$\overline{\gamma^i} = -\beta\gamma^i\beta$$

which is:

$$\overline{\gamma^0} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$\overline{\gamma^i} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

Doing the multiplication, we have:

$$\overline{\gamma^0} = \gamma^0$$

$$\overline{\gamma^i} = \gamma^i$$

as expected.

□

Next for $S^{\mu\nu}$:

$$\overline{S^{\mu\nu}} = \beta(S^{\mu\nu})^\dagger\beta$$

We have:

$$S^{\mu\nu} = \frac{i}{4}\begin{pmatrix} \sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu & 0 \\ 0 & \overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu \end{pmatrix}$$

The Pauli matrices (and the identity matrix) are Hermitian, so:

$$(S^{\mu\nu})^\dagger = -\frac{i}{4}\begin{pmatrix} \overline{\sigma}^\nu\sigma^\mu - \overline{\sigma}^\mu\sigma^\nu & 0 \\ 0 & \sigma^\nu\overline{\sigma}^\mu - \sigma^\mu\overline{\sigma}^\nu \end{pmatrix}$$

Thus:

$$(S^{\mu\nu})^\dagger = \frac{i}{4}\begin{pmatrix} \overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu & 0 \\ 0 & \sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu \end{pmatrix}$$

$$\overline{S^{\mu\nu}} = \frac{i}{4}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} \overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu & 0 \\ 0 & \sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$\overline{S^{\mu\nu}} = \frac{i}{4}\begin{pmatrix} \sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu & 0 \\ 0 & \overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu \end{pmatrix}$$
$$\overline{S^{\mu\nu}} = S^{\mu\nu}$$

□

Next for $i\gamma^5$:
$$\overline{i\gamma^5} = (-i)\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$
Multiplying the matrices gives:
$$\overline{i\gamma^5} = i\gamma^5$$
as expected. Note that the i contributed a minus sign, so it follows that $\overline{\gamma^5} = -\gamma^5$.

□

Next for $\gamma^\mu\gamma_5$:
$$\overline{\gamma^\mu\gamma_5} = \beta\gamma_5^\dagger\gamma^{\mu\dagger}\beta$$
$\beta^2 = I$, so:
$$\overline{\gamma^\mu\gamma_5} = \beta\gamma_5^\dagger\beta\beta\gamma^{\mu\dagger}\beta$$
We showed above that $\overline{\gamma^\mu} = \gamma^\mu$, and that $\overline{\gamma^5} = -\gamma^5$. Thus:
$$\overline{\gamma^\mu\gamma_5} = -\gamma_5\gamma^\mu$$
Now, note that $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. The γ^μ matrix will commute with itself but anticommute with the other three. Thus $\{\gamma^\mu, \gamma^5\} = 0$, and:
$$\overline{\gamma^\mu\gamma_5} = \gamma^\mu\gamma_5$$
as expected.

□

Now for $\overline{i\gamma_5 S^{\mu\nu}}$. We have:
$$\overline{i\gamma_5 S^{\mu\nu}} = -i\beta\gamma_5\beta\beta S^{\mu\nu}\beta$$
where the two βs in the middle are allowed since $\beta^2 = I$. Then,
$$\overline{i\gamma_5 S^{\mu\nu}} = -i\beta\gamma_5\beta\beta S^{\mu\nu}\beta$$
which is (remember that the gamma matrices including γ_5 are Hermitian, so we can add or remove the dagger at will):
$$\overline{i\gamma_5 S^{\mu\nu}} = -i\overline{\gamma_5}\,\overline{S^{\mu\nu}}$$
The γ_5 introduces a negative sign as discussed above. We also showed above that $S^{\mu\nu}$ is unchanged under this operation. This gives:
$$\overline{i\gamma_5 S^{\mu\nu}} = i\gamma_5 S^{\mu\nu}$$

□

Srednicki 38.3. Verify equation 38.22.

Subtracting 38.20 from 38.19:

$$\gamma^\mu \not{p} - \not{p}'\gamma^\mu = -p^\mu - 2iS^{\mu\nu}p_\nu + p'^\mu - 2iS^{\mu\nu}p'_v$$

We sandwich both sides between $\overline{u}_{s'}(p)$ and $v_s(-p)$ (we will neglect the spinor indices until the end; they are irrelevant to the problem):

$$\overline{u}(p)\left[\gamma^\mu \not{p} - \not{p}'\gamma^\mu\right]v(-p) = \overline{u}(p)\left[-p^\mu - 2iS^{\mu\nu}p_\nu + p'^\mu - 2iS^{\mu\nu}p'_v\right]v(-p)$$

Using equation 38.1:

$$\overline{u}(p)\left[-m\gamma^\mu - \not{p}'\gamma^\mu\right]v(-p) = \overline{u}(p)\left[-p^\mu - 2iS^{\mu\nu}p_\nu + p'^\mu - 2iS^{\mu\nu}p'_v\right]v(-p)$$

Next note that equation 38.16 gives:

$$\not{p}v(p) = mv(p)$$

$$\implies (-\not{p})v(-p) = mv(-p)$$

$$\implies \not{p}v(-p) = -mv(-p) \tag{38.3.1}$$

Hence,

$$\overline{u}(p)\left[-m\gamma^\mu + m\gamma^\mu\right]v(-p) = \overline{u}(p)\left[-p^\mu - 2iS^{\mu\nu}p_\nu + p'^\mu - 2iS^{\mu\nu}p'_v\right]v(-p)$$

This gives:

$$\overline{u}(p)\left[-p^\mu - 2iS^{\mu\nu}p_\nu + p'^\mu - 2iS^{\mu\nu}p'_v\right]v(-p) = 0$$

Next we take $p = p'$:

$$\overline{u}(p)\left[-4iS^{\mu\nu}p_\nu\right]v(-p) = 0$$

Dropping the constants:

$$\overline{u}(p)\left[S^{\mu\nu}p_\nu\right]v(-p) = 0$$

Using 36.53:

$$\overline{u}(p)\left[\begin{pmatrix} -\sigma^\nu & 0 \\ 0 & \sigma^\nu \end{pmatrix} p_\nu\right]v(-p) = 0$$

We can write this as:

$$\overline{u}(p)\left[\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\nu \\ -\sigma^\nu & 0 \end{pmatrix} p_\nu\right]v(-p) = 0$$

These are gamma matrices:

$$\overline{u}(p)\left[\gamma^0\gamma^\nu p_\nu\right]v(-p) = 0$$

Using equation 37.24:

$$\overline{u}(p)\gamma^0 \not{p}v(-p) = 0$$

Using equation (38.3.1):

$$-\overline{u}(p)\gamma^0 mv(-p) = 0$$

Dropping the constants:

$$\overline{u}_{s'}(p)\gamma^0 v_s(-p) = 0$$

$\square$

Taking the Hermitian conjugate:

$$\left[\overline{u}(p)\gamma^0 v(-p)\right]^\dagger = 0$$

This gives:

$$v(-p)^\dagger \gamma^{0\dagger} \overline{u}(p)^\dagger = 0$$

γ^0 is Hermitian. On the other terms, we use 38.7 (multiplying on the right by β). The result is:

$$\overline{v}(-p)\beta\gamma^0\beta u(p) = 0$$

Recall that β and γ^0 are the same (we keep them separate only to maintain the integrity of the indices). It is therefore obvious that they commute. Thus

$$\overline{v}(-p)\beta\beta\gamma^0 u(p) = 0$$

$\beta^2 = I$, so, reinserting the indices:

$$\overline{v}_s(-p)\gamma^0 u_{s'}(p) = 0$$

Finally, we let $p \to -p$ and $s \leftrightarrow s'$:

$$\overline{v}_{s'}(p)\gamma^0 u_s(-p) = 0$$

$\square$

Srednicki 38.4. Derive the Gordon identities:

$$\boldsymbol{\overline{u}_{s'}(p')\left[(p'+p)^\mu - 2iS^{\mu\nu}(p'-p)_\nu\right]\gamma_5 u_s(p) = 0}$$

$$\boldsymbol{\overline{v}_{s'}(p')\left[(p'+p)^\mu - 2iS^{\mu\nu}(p'-p)_\nu\right]\gamma_5 v_s(p) = 0}$$

The technique here is the same as in the previous problem. We start by adding 38.19 and 38.20:

$$\gamma^\mu \not{p} + \not{p}'\gamma^\mu = -p^\mu - p'^\mu - 2iS^{\mu\nu}p_\nu + 2iS^{\mu\nu}p'_\nu$$

We multiply on the right by γ_5, then sandwich between $\overline{u}(p')$ and $u(p)$. Thus:

$$\overline{u}(p')\left[\gamma^\mu \not{p} + \not{p}'\gamma^\mu\right]\gamma_5 u(p) = \overline{u}(p')\left[-p^\mu - p'^\mu - 2iS^{\mu\nu}p_\nu + 2iS^{\mu\nu}p'_\nu\right]\gamma_5 u(p)$$

Using equation 38.16:

$$\overline{u}(p')\left[\gamma^\mu \not{p} - m\gamma^\mu\right]\gamma_5 u(p) = -\overline{u}(p')\left[p^\mu + p'^\mu + 2iS^{\mu\nu}(p_\nu - p'_\nu)\right]\gamma_5 u(p)$$

Recall that γ^5 consists of four gamma matrices: γ^μ will anticommute with three of them and commute with one (itself):

$$\overline{u}(p')\left[-\gamma^\mu\gamma_5 \not{p} - m\gamma^\mu\gamma_5\right]\gamma_5 u(p) = -\overline{u}(p')\left[p^\mu + p'^\mu + 2iS^{\mu\nu}(p_\nu - p'_\nu)\right]\gamma_5 u(p)$$

Using 38.1:

$$\overline{u}(p')\left[m\gamma^\mu\gamma_5 - m\gamma^\mu\gamma_5\right]\gamma_5 u(p) = -\overline{u}(p')\left[p^\mu + p'^\mu + 2iS^{\mu\nu}(p_\nu - p'_\nu)\right]\gamma_5 u(p)$$

This left side vanishes:

$$-\overline{u}(p')\left[p^\mu + p'^\mu + 2iS^{\mu\nu}(p_\nu - p'_\nu)\right]\gamma_5 u(p) = 0$$

Dropping the constant and reordering:

$$\overline{u}(p')\left[p^\mu + p'^\mu - 2iS^{\mu\nu}(p' - p)_\nu\right]\gamma_5 u(p) = 0$$

$\square$

For the remaining identity, we repeat this with $u \leftrightarrow v$. We see from 38.1 and 38.16 that the result will be the same up to a minus sign; since there are two minus signs, there is no net result. Thus:

$$\overline{v}(p')\left[p^\mu + p'^\mu - 2iS^{\mu\nu}(p' - p)_\nu\right]\gamma_5 v(p) = 0$$

$\square$