

Srednicki Chapter 36

QFT Problems & Solutions

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Srednicki 36.1. Using the results of problem 2.9, show that, for a rotation by an angle θ about the z axis, we have:

$$D(\Lambda) = \exp(-i\theta S^{12})$$

and that, for a boost by rapidity η in the z direction, we have

$$D(\Lambda) = \exp(-i\eta S^{30})$$

In problem 2.9, we showed this is true for the vector representation. It must therefore be true for all other representations, including this spinor representation.

Srednicki 36.2. Verify that equation 36.46 is consistent with equation 36.43.

Equation 36.46 is:

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

Using equation 36.39:

$$\gamma_5 = i \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}$$

Doing the multiplication:

$$\gamma_5 = i \begin{pmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \begin{pmatrix} -\sigma_2\sigma_3 & 0 \\ 0 & -\sigma_2\sigma_3 \end{pmatrix}$$

This gives:

$$\gamma_5 = i \begin{pmatrix} \sigma_1\sigma_2\sigma_3 & 0 \\ 0 & -\sigma_1\sigma_2\sigma_3 \end{pmatrix}$$

Now recall that $-i\sigma_1\sigma_2\sigma_3 = I$. Then:

$$\gamma_5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

which is equation 36.43.

Srednicki 36.3.

(a) Prove the Fierz Identities

$$\left(\chi_1^\dagger \bar{\sigma}^\mu \chi_2\right) \left(\chi_3^\dagger \bar{\sigma}_\mu \chi_4\right) = -2\left(\chi_1^\dagger \chi_3^\dagger\right) \left(\chi_2 \chi_4\right)$$

$$\left(\chi_1^\dagger \bar{\sigma}^\mu \chi_2\right) \left(\chi_3^\dagger \bar{\sigma}_\mu \chi_4\right) = \left(\chi_1^\dagger \bar{\sigma}^\mu \chi_4\right) \left(\chi_3^\dagger \bar{\sigma}_\mu \chi_2\right)$$

Recall that the right-handed fields are always written as Hermitian Conjugates of left-handed fields. Thus, we insert the indices:

$$\left(\chi_1^\dagger \bar{\sigma}^\mu \chi_2\right) \left(\chi_3^\dagger \bar{\sigma}_\mu \chi_4\right) = \left(\chi_{1\dot{a}}^\dagger \bar{\sigma}^{\mu\dot{a}a} \chi_{2a}\right) \left(\chi_{3\dot{c}}^\dagger \bar{\sigma}_\mu^{\dot{c}c} \chi_4^c\right)$$

This is index notation, so we can move these around as we like:

$$\left(\chi_1^\dagger \bar{\sigma}^\mu \chi_2\right) \left(\chi_3^\dagger \bar{\sigma}_\mu \chi_4\right) = \bar{\sigma}_\mu^{\dot{c}c} \bar{\sigma}^{\mu\dot{a}a} \chi_{1\dot{a}}^\dagger \chi_{2a} \chi_{3\dot{c}}^\dagger \chi_4^c$$

Now we use equation 35.4:

$$\left(\chi_1^\dagger \bar{\sigma}^\mu \chi_2\right) \left(\chi_3^\dagger \bar{\sigma}_\mu \chi_4\right) = -2\varepsilon^{ac} \varepsilon^{\dot{a}\dot{c}} \chi_{1\dot{a}}^\dagger \chi_{2a} \chi_{3\dot{c}}^\dagger \chi_4^c$$

Now we use the Levi-Cevita symbol to raise the indices (recall that the Levi-Cevita symbol is the spinor analog of the metric). We also move them around at will. Then:

$$\left(\chi_1^\dagger \bar{\sigma}^\mu \chi_2\right) \left(\chi_3^\dagger \bar{\sigma}_\mu \chi_4\right) = -2\chi_{1\dot{c}}^\dagger \chi_{3\dot{c}} \chi_{2a} \chi_4^a$$

Now we drop the indices since this are just multiplied:

$$\left(\chi_1^\dagger \bar{\sigma}^\mu \chi_2\right) \left(\chi_3^\dagger \bar{\sigma}_\mu \chi_4\right) = -2\left(\chi_1^\dagger \chi_3^\dagger\right) \left(\chi_2 \chi_4\right)$$

Now we want to get back where we started with $2 \leftrightarrow 4$. We know that $\chi_2 \chi_4 = \chi_4 \chi_2$. [If you're confused about why the fields seem to commute when fermions should obviously anticommute, see equation 35.25.] Then:

$$\left(\chi_1^\dagger \bar{\sigma}^\mu \chi_2\right) \left(\chi_3^\dagger \bar{\sigma}_\mu \chi_4\right) = -2\left(\chi_1^\dagger \chi_3^\dagger\right) \left(\chi_4 \chi_2\right)$$

This shows that the right hand side of equation 36.58 is invariant under this relabeling. It follows that the left hand side of equation 36.58 is also invariant under this relabeling – which is exactly what equation 36.59 claims.

(b) Define the Dirac fields

$$\Psi_i = \begin{pmatrix} \chi_i \\ \xi_i^\dagger \end{pmatrix}, \quad \Psi_i^C = \begin{pmatrix} \xi_i \\ \chi_i^\dagger \end{pmatrix}$$

Use equations 36.58 and 36.59 to prove the Dirac Form of the Fierz identities,

$$\left(\bar{\Psi}_1 \gamma^\mu P_L \Psi_2\right) \left(\bar{\Psi}_3 \gamma^\mu P_L \Psi_4\right) = -2\left(\bar{\Psi}_1 P_R \Psi_3^C\right) \left(\bar{\Psi}_4^C P_L \Psi_2\right)$$

$$(\bar{\Psi}_1 \gamma^\mu P_L \Psi_2) (\bar{\Psi}_3 \gamma^\mu P_L \Psi_4) = (\bar{\Psi}_1 \gamma^\mu P_L \Psi_4) (\bar{\Psi}_3 \gamma^\mu P_L \Psi_2)$$

All we actually have to do is prove that these forms of the Fierz Identities are equivalent to those in part (a). Let's start with equation 36.58:

$$(\chi_1^\dagger \bar{\sigma}^\mu \chi_2) (\chi_3^\dagger \bar{\sigma}_\mu \chi_4) = -2(\chi_1^\dagger \chi_3^\dagger) (\chi_2 \chi_4)$$

On both sides, let's write this as a bra times a ket.

$$\left[(\xi_1, \chi_1^\dagger) \begin{pmatrix} 0 \\ \bar{\sigma}^\mu \chi_2 \end{pmatrix} \right] \left[(\xi_3, \chi_3^\dagger) \begin{pmatrix} 0 \\ \bar{\sigma}_\mu \chi_4 \end{pmatrix} \right] = -2 \left[(\xi_1, \chi_1^\dagger) \begin{pmatrix} 0 \\ \chi_3^\dagger \end{pmatrix} \right] \left[(\chi_4, \xi_4^\dagger) \begin{pmatrix} \chi_2 \\ 0 \end{pmatrix} \right]$$

where in the last term we remember, as in part (a), that $\chi_2 \chi_4 = \chi_4 \chi_2$. We can use equation 36.60 to identify some of these terms:

$$\left[\bar{\Psi}_1 \begin{pmatrix} 0 \\ \bar{\sigma}^\mu \chi_2 \end{pmatrix} \right] \left[\bar{\Psi}_3 \begin{pmatrix} 0 \\ \bar{\sigma}_\mu \chi_4 \end{pmatrix} \right] = -2 \left[\bar{\Psi}_1 \begin{pmatrix} 0 \\ \chi_3^\dagger \end{pmatrix} \right] \left[\bar{\Psi}_4^C \begin{pmatrix} \chi_2 \\ 0 \end{pmatrix} \right]$$

On the left-hand side, let's separate the σ s into their own matrix:

$$\left[\bar{\Psi}_1 \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} \chi_2 \\ 0 \end{pmatrix} \right] \left[\bar{\Psi}_3 \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix} \begin{pmatrix} \chi_4 \\ 0 \end{pmatrix} \right] = -2 \left[\bar{\Psi}_1 \begin{pmatrix} 0 \\ \chi_3^\dagger \end{pmatrix} \right] \left[\bar{\Psi}_4^C \begin{pmatrix} \chi_2 \\ 0 \end{pmatrix} \right]$$

These are gamma matrices:

$$\left[\bar{\Psi}_1 \gamma^\mu \begin{pmatrix} \chi_2 \\ 0 \end{pmatrix} \right] \left[\bar{\Psi}_3 \gamma_\mu \begin{pmatrix} \chi_4 \\ 0 \end{pmatrix} \right] = -2 \left[\bar{\Psi}_1 \begin{pmatrix} 0 \\ \chi_3^\dagger \end{pmatrix} \right] \left[\bar{\Psi}_4^C \begin{pmatrix} \chi_2 \\ 0 \end{pmatrix} \right]$$

Now let's rewrite the remaining kets as projections:

$$\left[\bar{\Psi}_1 \gamma^\mu P_L \begin{pmatrix} \chi_2 \\ \xi_2^\dagger \end{pmatrix} \right] \left[\bar{\Psi}_3 \gamma_\mu P_L \begin{pmatrix} \chi_4 \\ \xi_4^\dagger \end{pmatrix} \right] = -2 \left[\bar{\Psi}_1 P_R \begin{pmatrix} \xi_3 \\ \chi_3^\dagger \end{pmatrix} \right] \left[\bar{\Psi}_4^C P_L \begin{pmatrix} \chi_2 \\ \xi_2^\dagger \end{pmatrix} \right]$$

Using 36.60 again:

$$\left[\bar{\Psi}_1 \gamma^\mu P_L \Psi_2 \right] \left[\bar{\Psi}_3 \gamma_\mu P_L \Psi_4 \right] = -2 \left[\bar{\Psi}_1 P_R \Psi_3^C \right] \left[\bar{\Psi}_4^C P_L \Psi_2 \right] \quad (36.3.1)$$

which is 36.61.

To prove 36.62, let's go back to equation 36.58 and write it as a bra times a ket as before, but this time we will write the last term differently:

$$\left[(\xi_1, \chi_1^\dagger) \begin{pmatrix} 0 \\ \bar{\sigma}^\mu \chi_2 \end{pmatrix} \right] \left[(\xi_3, \chi_3^\dagger) \begin{pmatrix} 0 \\ \bar{\sigma}_\mu \chi_4 \end{pmatrix} \right] = -2 \left[(\xi_1, \chi_1^\dagger) \begin{pmatrix} 0 \\ \chi_3^\dagger \end{pmatrix} \right] \left[(\chi_2, \xi_2^\dagger) \begin{pmatrix} \chi_4 \\ 0 \end{pmatrix} \right]$$

The left-hand side has not changed; the right hand side is the same up to an arbitrary relabeling ($2 \leftrightarrow 4$). We can therefore read off the result from equation (36.3.1):

$$\left[\bar{\Psi}_1 \gamma^\mu P_L \Psi_2 \right] \left[\bar{\Psi}_3 \gamma_\mu P_L \Psi_4 \right] = -2 \left[\bar{\Psi}_1 P_R \Psi_3^C \right] \left[\bar{\Psi}_2^C P_L \Psi_4 \right]$$

This shows that the right-hand side of equation 36.61 is invariant under $2 \leftrightarrow 4$. It follows that the left-hand side must also be invariant under that transformation, ie that:

$$[\bar{\Psi}_1 \gamma^\mu P_L \Psi_2] [\bar{\Psi}_3 \gamma_\mu P_L \Psi_4] = [\bar{\Psi}_1 \gamma^\mu P_L \Psi_4] [\bar{\Psi}_3 \gamma_\mu P_L \Psi_2]$$

which is equation 36.62.

Note: This may seem like a brilliant but completely unintuitive solution. If so, just start with the result, apply the projection and gamma matrices, and arrive at 36.58-59. Then reorder your results to get this solution (or announce that all your steps are reversible, so you're already done).

(c) By writing both sides out in terms of Weyl fields, show that

$$\begin{aligned}\bar{\Psi}_1 \gamma^\mu P_R \Psi_2 &= -\bar{\Psi}_2^C \gamma^\mu P_L \Psi_1^C \\ \bar{\Psi}_1 P_L \Psi_2 &= \bar{\Psi}_2^C P_L \Psi_1^C \\ \bar{\Psi}_1 P_R \Psi_2 &= \bar{\Psi}_2^C P_R \Psi_1^C\end{aligned}$$

Writing both sides as instructed, we have:

$$\begin{pmatrix} \xi_1 \\ \chi_1^\dagger \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \xi_2^\dagger \end{pmatrix} \stackrel{?}{=} -(\chi_2^\dagger \xi_2) \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ 0 \end{pmatrix}$$

Doing the matrix multiplication:

$$\xi_1 \sigma^\mu \xi_2^\dagger \stackrel{?}{=} -\xi_2^\dagger \bar{\sigma}^\mu \xi_1$$

To see if this can be true, let's insert the indices. As usual, the daggers represent right-handed fields, the non-daggered fields are left handed (that's our convention). We give dotted indices to right-handed fields, non-dotted indices to left-handed fields. Further, $\bar{\sigma}$ takes raised indices, σ takes lowered indices. Then:

$$\xi_1^a \sigma_{a\dot{a}}^\mu \xi_2^{\dagger\dot{a}} \stackrel{?}{=} -\xi_2^{\dagger\dot{a}} \bar{\sigma}^{\mu\dot{a}a} \xi_{1a}$$

On the left-hand side, let's use the Levi-Cevita symbol to raise our indices. Remember we have to contract with the second index of the Levi-Cevita symbol, otherwise we get a minus sign. Then,

$$\varepsilon^{ab} \varepsilon^{\dot{a}\dot{b}} \xi_{1b} \sigma_{a\dot{a}}^\mu \xi_2^{\dagger\dot{b}} \stackrel{?}{=} -\xi_2^{\dagger\dot{a}} \bar{\sigma}^{\mu\dot{a}a} \xi_{1a}$$

Now let's use equaton 35.19 (which is a definition):

$$\xi_{1b} \bar{\sigma}^{\mu\dot{b}b} \xi_2^{\dagger\dot{b}} \stackrel{?}{=} -\xi_2^{\dagger\dot{a}} \bar{\sigma}^{\mu\dot{a}a} \xi_{1a}$$

In order to be able to drop the indices on the left-hand side, we need to reverse the order of the two terms. However, note that the function indicated by the index of a Weyl Field is a fermionic function, and these always anticommute. Thus,

$$-\xi_2^{\dagger\dot{a}} \bar{\sigma}^{\mu\dot{b}b} \xi_{1b} \stackrel{?}{=} -\xi_2^{\dagger\dot{a}} \bar{\sigma}^{\mu\dot{a}a} \xi_{1a}$$

Dropping the indices, we find:

$$-\xi_2^\dagger \bar{\sigma}^\mu \xi_1 \stackrel{?}{=} -\xi_2^\dagger \bar{\sigma}^\mu \xi_1$$

□

Expanding the second equation, we have:

$$(\xi_1, \chi_1^\dagger) \begin{pmatrix} \chi_2 \\ 0 \end{pmatrix} \stackrel{?}{=} (\chi_2 \xi_2^\dagger) \begin{pmatrix} \xi_1 \\ 0 \end{pmatrix}$$

This gives:

$$\xi_1 \chi_2 \stackrel{?}{=} \chi_2 \xi_1$$

□

The third equation is evaluated in the same manner as the second; the result is

$$\chi_1^\dagger \xi_2^\dagger = \xi_2^\dagger \chi_1^\dagger$$

by is true (see equation 35.25).

□

Srednicki 36.4. Consider a field $\phi_A(x)$ in an unspecified representation of the Lorentz Group, indexed by A, that obeys

$$U(\Lambda)^{-1} \phi_A(x) U(\Lambda) = L_A^B(\Lambda) \phi_B(\Lambda^{-1}x)$$

For an infinitesimal transformation:

$$L_A^B(1 + \delta\omega) = \delta_A^B + \frac{i}{2} \delta\omega_{\mu\nu} (S^{\mu\nu})_A^B$$

(a) Following the procedure of section 22, show that the energy-momentum tensor is

$$T^{\mu\nu} = g^{\mu\nu} \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_A)} \partial^\nu \phi_A$$

The derivation in chapter 22 still holds: everything is the same except that we've replaced the scalar field ϕ_a with the representation-independent field ϕ_A . We therefore read off the energy-momentum tensor from equation 22.29; the result is equation 36.68.

(b) Show that the Noether current corresponding to a Lorentz transformation is

$$\mathcal{M}^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} + B^{\mu\nu\rho}$$

where

$$B^{\mu\nu\rho} = -i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_A)} (S^{\nu\rho})_A^B \phi_B$$

Equation 22.27 gives the Noether Current:

$$j^\mu(x) = \frac{\partial \mathcal{L}(x)}{\partial(\partial_\mu \phi_A(x))} \delta \phi_A(x) - K^\mu(x) \quad (36.4.1)$$

We need $\delta\phi_A$, which we get from the symmetry. Our Lorentz Symmetry is, in differential form:

$$\phi_A \rightarrow L_A^B(1 + \delta\omega)\phi_B(x^\mu - x_\nu\delta_\omega^{\mu\nu})$$

We use 36.67 for L_A^B and make a Taylor Series of ϕ_B . Then:

$$\phi_A \rightarrow \left[\delta_A^B + \frac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu})_A^B \right] [\phi_B(x) - \partial^\mu\phi_A x^\nu\delta\omega_{\mu\nu}]$$

Doing the multiplication:

$$\phi_A \rightarrow \phi_A - \partial^\mu\phi_A x^\nu\delta\omega_{\mu\nu} + \frac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu})_A^B\phi_B$$

Hence:

$$\delta\phi = \left[\frac{i}{2}(S^{\mu\nu})_A^B\phi_B - \partial^\mu\phi_A x^\nu \right] \delta\omega_{\mu\nu} \quad (36.4.2)$$

Now for K^μ . To determine this, we need to look at the change in the Lagrangian:

$$\mathcal{L}(x) \rightarrow \mathcal{L}(\Lambda^{-1}x)$$

$$\mathcal{L}(x) \rightarrow \mathcal{L}(x^\mu - \delta\omega^{\mu\nu}x_\nu)$$

$$\mathcal{L}(x) \rightarrow \mathcal{L}(x) - \partial^\mu\mathcal{L}\delta\omega_{\mu\nu}x^\nu$$

Therefore:

$$\delta\mathcal{L} = -\partial^\mu\mathcal{L}\delta\omega_{\mu\nu}x^\nu$$

We can write this as:

$$\delta\mathcal{L} = -\partial^\mu(x^\nu\mathcal{L})\delta\omega_{\mu\nu}$$

Why? We can recover the original equation by doing the derivative with the product rule. One derivative will give a metric, which we use to change the differential to $\delta\omega_\mu^\mu$. This represents the trace of an antisymmetric matrix, which vanishes. The other term is the original equation.

Next, we bring $\delta\omega$ inside the derivative as well. We can treat this as a constant term because it is a differential: taking the derivative of a differential will effectively give a second-order differential, which we neglect. Then,

$$\delta\mathcal{L} = -\partial^\mu(x^\nu\mathcal{L}\delta\omega_{\mu\nu})$$

Next we swap the indices:

$$\delta\mathcal{L} = -\partial_\mu(x_\nu\mathcal{L}\delta\omega^{\mu\nu})$$

which gives, by the definition of K^μ :

$$K^\mu = -\delta\omega^{\mu\nu}x_\nu\mathcal{L} \quad (36.4.3)$$

Finally, we combine (36.4.2) and (36.4.3) into (36.4.1), the result is (changing the indices as needed):

$$j^\rho = \frac{\partial\mathcal{L}(x)}{\partial(\partial_\rho\phi_A(x))} \left[\frac{i}{2}(S^{\mu\nu})_A^B\phi_B - \partial^\mu\phi_A x^\nu \right] \delta\omega_{\mu\nu} + \mathcal{L}\delta\omega^{\rho\nu}x_\nu$$

We distribute the first term:

$$j^\rho = \frac{\partial \mathcal{L}(x)}{\partial(\partial_\rho \phi_A(x))} \frac{i}{2} (S^{\mu\nu})_A^B \phi_B - \partial^\mu \phi_A x^\nu \delta\omega_{\mu\nu} \frac{\partial \mathcal{L}(x)}{\partial(\partial_\rho \phi_A(x))} + \mathcal{L} \delta\omega^{\rho\nu} x_\nu$$

This first term is B:

$$j^\rho = -\frac{1}{2} B^{\rho\mu\nu} \delta\omega_{\mu\nu} - \partial^\mu \phi_A x^\nu \delta\omega_{\mu\nu} \frac{\partial \mathcal{L}(x)}{\partial(\partial_\rho \phi_A(x))} + \mathcal{L} \delta\omega^{\rho\nu} x_\nu$$

Using the result of part (a), we have:

$$j^\rho = -\frac{1}{2} B^{\rho\mu\nu} \delta\omega_{\mu\nu} + T^{\rho\mu} x^\nu \delta\omega_{\mu\nu} - g^{\rho\mu} \mathcal{L} x^\nu \delta\omega_{\mu\nu} + \mathcal{L} \delta\omega^{\rho\nu} x_\nu$$

After manipulating the indices, these last two terms cancel:

$$j^\rho = -\frac{1}{2} B^{\rho\mu\nu} \delta\omega_{\mu\nu} + T^{\rho\mu} x^\nu \delta\omega_{\mu\nu}$$

Rewriting this:

$$j^\rho = -\frac{1}{2} [B^{\rho\mu\nu} \delta\omega_{\mu\nu} - T^{\rho\mu} x^\nu \delta\omega_{\mu\nu} - T^{\rho\mu} x^\nu \delta\omega_{\mu\nu}]$$

In this last term, μ and ν are just dummy indices. We'll exchange them:

$$j^\rho = -\frac{1}{2} [B^{\rho\mu\nu} \delta\omega_{\mu\nu} - T^{\rho\mu} x^\nu \delta\omega_{\mu\nu} - T^{\rho\nu} x^\mu \delta\omega_{\nu\mu}]$$

$\delta\omega$ is antisymmetric, so we can switch those indices:

$$j^\rho = -\frac{1}{2} [B^{\rho\mu\nu} \delta\omega_{\mu\nu} - T^{\rho\mu} x^\nu \delta\omega_{\mu\nu} + T^{\rho\nu} x^\mu \delta\omega_{\mu\nu}]$$

Now we factor out the differential:

$$j^\rho = -\frac{1}{2} [B^{\rho\mu\nu} - T^{\rho\mu} x^\nu + T^{\rho\nu} x^\mu] \delta\omega_{\mu\nu}$$

We recognize the term in the brackets as \mathcal{M} :

$$j^\rho = -\frac{1}{2} [\mathcal{M}^{\rho\mu\nu}] \delta\omega_{\mu\nu}$$

It is customary to drop the differential as well as the constant terms. Then,

$$j^{\rho\mu\nu} = \mathcal{M}^{\rho\mu\nu}$$

as expected.

(c) Use the conservation laws $\partial_\mu T^{\mu\nu} = 0$ and $\partial_\mu \mathcal{M}^{\mu\nu\rho} = 0$ to show that

$$T^{\nu\rho} - T^{\rho\nu} + \partial_\mu B^{\mu\nu\rho} = 0$$

We have:

$$\partial_\mu \mathcal{M}^{\mu\nu\rho} = \partial_\mu (x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} + B^{\mu\nu\rho}) = 0$$

This gives:

$$\partial_\mu \mathcal{M}^{\mu\nu\rho} = g_\mu^\nu T^{\mu\rho} + x^\nu \partial_\mu T^{\mu\rho} - g_\mu^\rho T^{\mu\nu} - x^\rho \partial_\mu T^{\mu\nu} + \partial_\mu B^{\mu\nu\rho} = 0$$

The derivative of T is zero, so:

$$T^{\nu\rho} - T^{\rho\nu} + \partial_\mu B^{\mu\nu\rho} = 0$$

(d) Define the improved energy-momentum tensor or Belinfante tensor

$$\Theta^{\mu\nu} = T^{\mu\nu} + \frac{1}{2} \partial_\rho (B^{\rho\mu\nu} - B^{\mu\rho\nu} - B^{\nu\rho\mu})$$

(i) Show that $\Theta^{\mu\nu}$ is symmetric.

We have:

$$\Theta^{\mu\nu} = T^{\mu\nu} + \frac{1}{2} \partial_\rho (B^{\rho\mu\nu} - B^{\mu\rho\nu} - B^{\nu\rho\mu})$$

We can simply reorder these last two terms:

$$\Theta^{\mu\nu} = T^{\mu\nu} + \frac{1}{2} \partial_\rho (B^{\rho\mu\nu} - B^{\nu\rho\mu} - B^{\mu\rho\nu})$$

From the definition of B, we see that the last two indices of B go onto $S^{\mu\nu}$, the generators of the Lorentz Group for spinors. Since S is antisymmetric, it follows that B is antisymmetric in its last two indices. Therefore, we can write:

$$\Theta^{\mu\nu} = T^{\mu\nu} + \frac{1}{2} \partial_\rho (-B^{\rho\nu\mu} - B^{\nu\rho\mu} - B^{\mu\rho\nu})$$

We can even rewrite this as:

$$\Theta^{\mu\nu} = T^{\mu\nu} - \partial_\rho B^{\rho\nu\mu} + \frac{1}{2} \partial_\rho (B^{\rho\nu\mu} - B^{\nu\rho\mu} - B^{\mu\rho\nu})$$

Again using the fact that B is antisymmetric in its last two indices, we write:

$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\rho B^{\rho\mu\nu} + \frac{1}{2} \partial_\rho (B^{\rho\nu\mu} - B^{\nu\rho\mu} - B^{\mu\rho\nu})$$

Using the result from part (c), we have:

$$\Theta^{\mu\nu} = T^{\nu\mu} + \frac{1}{2} \partial_\rho (B^{\rho\nu\mu} - B^{\nu\rho\mu} - B^{\mu\rho\nu})$$

which gives:

$$\Theta^{\mu\nu} = \Theta^{\nu\mu}$$

(ii) Show that $\Theta^{\mu\nu}$ is conserved, $\partial_\mu \Theta^{\mu\nu} = 0$

We take the derivative:

$$\partial_\mu \Theta^{\mu\nu} = \partial_\mu T^{\mu\nu} + \frac{1}{2} \partial_\mu \partial_\rho (B^{\rho\mu\nu} - B^{\mu\rho\nu} - B^{\nu\rho\mu})$$

This first term vanishes by Conservation of Energy. In the remaining terms, we will distribute:

$$\partial_\mu \Theta^{\mu\nu} = \frac{1}{2} \partial_\mu \partial_\rho B^{\rho\mu\nu} - \frac{1}{2} \partial_\mu \partial_\rho B^{\mu\rho\nu} - \frac{1}{2} \partial_\mu \partial_\rho B^{\nu\rho\mu}$$

For the second term, we will use our usual trick of arbitrarily choosing to swap the dummy indices ρ and ν . Also, we'll divide the last term into two. These two changes give:

$$\partial_\mu \Theta^{\mu\nu} = \frac{1}{2} \partial_\mu \partial_\rho B^{\rho\mu\nu} - \frac{1}{2} \partial_\rho \partial_\mu B^{\rho\mu\nu} - \frac{1}{4} \partial_\mu \partial_\rho B^{\nu\rho\mu} - \frac{1}{4} \partial_\mu \partial_\rho B^{\nu\rho\mu}$$

Recall that partial derivatives commute; thus, the first and second terms cancel. In the fourth term, we'll again use our trick of swapping the dummy indices. Then:

$$\partial_\mu \Theta^{\mu\nu} = -\frac{1}{4} \partial_\mu \partial_\rho B^{\nu\rho\mu} - \frac{1}{4} \partial_\rho \partial_\mu B^{\nu\mu\rho}$$

Now in the fourth term, we can again commute the partial derivatives. In addition, we have already discussed how B is antisymmetric in its last two indices. Thus, the remaining two terms cancel:

$$\partial_\mu \Theta^{\mu\nu} = 0$$

(iii) Show that $\int d^3x \Theta^{0\nu} = \int d^3x T^{0\nu} = P^\nu$, where P^ν is the energy-momentum four-vector. In general relativity, it is the Belinfante tensor that couples to gravity.

We have:

$$\int \Theta^{0\nu} d^3x = \int T^{0\nu} d^3x + \frac{1}{2} \int \partial_\rho (B^{\rho 0\nu} - B^{0\rho\nu} - B^{\nu\rho 0}) d^3x$$

We break this last term into temporal and spatial counterparts:

$$\int \Theta^{0\nu} d^3x = \int T^{0\nu} d^3x + \frac{1}{2} \int \partial_i (B^{i 0\nu} - B^{0i\nu} - B^{\nu i 0}) d^3x + \frac{1}{2} \int \partial_0 (B^{0 0\nu} - B^{0 0\nu} - B^{\nu 0 0}) d^3x$$

The second integral on the right hand side vanishes, as it is an integral over a total divergence (assuming reasonable boundary conditions at spatial infinity). In the last integral, the first two terms cancel, and the third term vanishes because B is antisymmetric in its last two indices, as discussed above. We're left with:

$$\int \Theta^{0\nu} d^3x = \int T^{0\nu} d^3x = P^\nu$$

where the last equality follows by equation 22.35.

(e) We define the improved tensor by:

$$\Xi^{\mu\nu\rho} = x^\nu \Theta^{\mu\rho} - x^\rho \Theta^{\mu\nu}$$

(i) Show that it obeys $\partial_\mu \Xi^{\mu\nu\rho} = 0$.

We evaluate the derivative:

$$\partial_\mu \Xi^{\mu\nu\rho} = \partial_\mu (x^\nu \Theta^{\mu\rho} - x^\rho \Theta^{\mu\nu})$$

Using the product rule:

$$\partial_\mu \Xi^{\mu\nu\rho} = (\partial_\mu x^\nu) \Theta^{\mu\rho} + x^\nu \partial_\mu \Theta^{\mu\rho} - (\partial_\mu x^\rho) \Theta^{\mu\nu} - x^\rho \partial_\mu \Theta^{\mu\nu}$$

Since the Belinfante tensor is conserved (our result from part (d)), the second and fourth terms vanish. In the remaining terms, we evaluate the derivative:

$$\partial_\mu \Xi^{\mu\nu\rho} = g_\mu^\nu \Theta^{\mu\rho} - g_\mu^\rho \Theta^{\mu\nu}$$

Using the metric:

$$\partial_\mu \Xi^{\mu\nu\rho} = \Theta^{\nu\rho} - \Theta^{\rho\nu}$$

Since the Belinfante Tensor is symmetric (another result from part (d)), we have:

$$\partial_\mu \Xi^{\mu\nu\rho} = 0$$

(ii) Show that $\int d^3x \Xi^{0\nu\rho} = \int d^3x \mathcal{M}^{0\nu\rho} = M^{\nu\rho}$, where $M^{\nu\rho}$ are the Lorentz generators.

We have:

$$\int \Xi^{0\nu\rho} = \int [x^\nu \Theta^{0\rho} - x^\rho \Theta^{0\nu}]$$

Using the definition of Θ , we have:

$$\int \Xi^{0\nu\rho} d^3x = \int \left[x^\nu \left(T^{0\rho} + \frac{1}{2} \partial_\mu [B^{\mu 0\rho} - B^{0\mu\rho} - B^{\rho\mu 0}] \right) - x^\rho \left(T^{0\nu} + \frac{1}{2} \partial_\mu [B^{\mu 0\nu} - B^{0\mu\nu} - B^{\nu\mu 0}] \right) \right] d^3x$$

Distributing:

$$\int \Xi^{0\nu\rho} d^3x = \int \left(x^\nu T^{0\rho} + \frac{1}{2} x^\nu \partial_\mu (B^{\mu 0\rho} - B^{0\mu\rho} - B^{\rho\mu 0}) - x^\rho T^{0\nu} - \frac{1}{2} x^\rho \partial_\mu (B^{\mu 0\nu} - B^{0\mu\nu} - B^{\nu\mu 0}) \right) d^3x$$

We break the sums over μ into temporal and spatial components:

$$\begin{aligned} \int \Xi^{0\nu\rho} d^3x = \int & \left(x^\nu T^{0\rho} + \frac{1}{2} x^\nu \partial_0 (B^{00\rho} - B^{00\rho} - B^{\rho 00}) + \frac{1}{2} x^\nu \partial_i (B^{i0\rho} - B^{0i\rho} - B^{\rho i0}) - x^\rho T^{0\nu} \right. \\ & \left. - \frac{1}{2} x^\rho \partial_0 (B^{00\nu} - B^{00\nu} - B^{\nu 00}) - \frac{1}{2} x^\rho \partial_i (B^{i0\nu} - B^{0i\nu} - B^{\nu i0}) \right) d^3x \end{aligned}$$

The temporal terms vanish as before. Then:

$$\int \Xi^{0\nu\rho} d^3x = \int \left(x^\nu T^{0\rho} + \frac{1}{2} x^\nu \partial_i (B^{i0\rho} - B^{0i\rho} - B^{\rho i0}) - x^\rho T^{0\nu} - \frac{1}{2} x^\rho \partial_i (B^{i0\nu} - B^{0i\nu} - B^{\nu i0}) \right) d^3x$$

Integrating by parts, we have:

$$\int \Xi^{0\nu\rho} d^3x = \int \left(x^\nu T^{0\rho} - \frac{1}{2} (\partial_i x^\nu) (B^{i0\rho} - B^{0i\rho} - B^{\rho i0}) - x^\rho T^{0\nu} + \frac{1}{2} (\partial_i x^\rho) (B^{i0\nu} - B^{0i\nu} - B^{\nu i0}) \right) d^3x$$

Doing the derivatives:

$$\int \Xi^{0\nu\rho} d^3x = \int \left(x^\nu T^{0\rho} - \frac{1}{2} g_i{}^\nu (B^{i0\rho} - B^{0i\rho} - B^{\rho i0}) - x^\rho T^{0\nu} + \frac{1}{2} g_i{}^\rho (B^{i0\nu} - B^{0i\nu} - B^{\nu i0}) \right) d^3x$$

Using the metric:

$$\int \Xi^{0\nu\rho} d^3x = \int \left(x^\nu T^{0\rho} - \frac{1}{2} (B^{\nu 0\rho} - B^{0\nu\rho} - B^{\rho\nu 0}) - x^\rho T^{0\nu} + \frac{1}{2} (B^{\rho 0\nu} - B^{0\rho\nu} - B^{\nu\rho 0}) \right) d^3x$$

The first and last terms cancel, as do the third and fourth. Then:

$$\int \Xi^{0\nu\rho} d^3x = \int [x^\nu T^{0\rho} - x^\rho T^{0\nu} + B^{0\nu\rho}] d^3x$$

which is:

$$\int \Xi^{0\nu\rho} d^3x = \int \mathcal{M}^{0\nu\rho} d^3x = M^{\nu\rho}$$

where the last equality follows by equation 22.40.

(f) Compute $\Theta^{\mu\nu}$ for a left-handed Weyl field with \mathcal{L} given by equation 36.2, and for a Dirac field with \mathcal{L} given by equation 36.28.

The Lagrangian given by equation 36.2 is:

$$\mathcal{L} = i\psi^\dagger \bar{\rho}^\rho \partial_\rho \psi - \frac{1}{2} m\psi\psi - \frac{1}{2} m\psi^\dagger \psi^\dagger$$

where the m has been made real and positive as per the discussion in the text. We've also changed the dummy index to ρ to avoid ambiguity later on. Then we can take the derivative:

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} = i\psi^\dagger \bar{\sigma}^\mu$$

Equation 36.68 gives, therefore:

$$T^{\mu\nu} = g^{\mu\nu} \left[i\psi^\dagger \bar{\sigma}^\rho \partial_\rho \psi - \frac{1}{2} m\psi\psi - \frac{1}{2} m\psi^\dagger \psi^\dagger \right] - i\psi^\dagger \bar{\sigma}^\mu \partial^\nu \psi$$

Now we note that, since this is a left-handed Weyl Field, we take the generator of the Lorentz Group by equation 36.51:

$$(S_L^{\mu\nu})_A^B = \frac{i}{4} [\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu]_A^B$$

Equation 36.70 gives, therefore:

$$B^{\mu\nu\rho} = \frac{i}{4} \psi^\dagger \bar{\sigma}^\mu [\sigma^\nu \bar{\sigma}^\rho - \sigma^\rho \bar{\sigma}^\nu]_A^B \psi_B$$

Combining all these results in equation 36.72, we have:

$$\Theta^{\mu\nu} = g^{\mu\nu} \left[i\psi^\dagger \bar{\sigma}^\rho \partial_\rho \psi - \frac{1}{2} m \psi \psi - \frac{1}{2} m \psi^\dagger \psi^\dagger \right] - i\psi^\dagger \bar{\sigma}^\mu \partial^\nu \psi + \frac{1}{2} \partial_\rho \left[\frac{i}{4} \psi^\dagger \bar{\sigma}^A \bar{\sigma}^\rho [\sigma^\mu \bar{\sigma}^\nu - \sigma^\rho \bar{\sigma}^\nu]_A^B \psi_B \right. \\ \left. - \frac{i}{4} \psi^\dagger \bar{\sigma}^A \bar{\sigma}^\mu [\sigma^\rho \bar{\sigma}^\nu - \sigma^\rho \bar{\sigma}^\nu]_A^B \psi_B - \frac{i}{4} \psi^\dagger \bar{\sigma}^\nu [\sigma^\rho \bar{\sigma}^\mu - \sigma^\rho \bar{\sigma}^\nu]_A^B \psi_B \right]$$

There's not much we can do to simplify this. We could commute the derivative through the $\bar{\sigma}$, but that would require us to use the product rule, making this more complicated rather than less so; further, nothing would cancel. All we can really do is to pull out some common factors:

$$\Theta^{\mu\nu} = i g^{\mu\nu} \psi^\dagger \bar{\sigma}^\rho \partial_\rho \psi - \frac{m}{2} g^{\mu\nu} \psi \psi - \frac{m}{2} g^{\mu\nu} \psi^\dagger \psi^\dagger - i\psi^\dagger \bar{\sigma}^\mu \partial^\nu \psi + \frac{i}{8} \partial_\rho \left[\psi^\dagger \bar{\sigma}^A \bar{\sigma}^\rho (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)_A^B \psi_B \right. \\ \left. - \psi^\dagger \bar{\sigma}^A \bar{\sigma}^\mu (\sigma^\rho \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\rho)_A^B \psi_B - \psi^\dagger \bar{\sigma}^\nu (\sigma^\rho \bar{\sigma}^\mu - \sigma^\mu \bar{\sigma}^\rho)_A^B \psi_B \right]$$

□

As for the Dirac Field, we again have:

$$\Theta^{\mu\nu} = T^{\mu\nu} - \frac{1}{2} \partial_\rho (B^{\rho\mu\nu} - B^{\mu\rho\nu} - B^{\nu\rho\mu}) \quad (36.4.4)$$

The energy-momentum tensor is:

$$T^{\mu\nu} = g^{\mu\nu} \mathcal{L} - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Psi)} \partial^\nu \Psi$$

The Lagrangian is given by equation 36.28:

$$\mathcal{L} = i \bar{\Psi} \gamma^\mu \partial_\mu \Psi - m \bar{\Psi} \Psi$$

and so we can compute

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Psi)} = i \bar{\Psi} \gamma^\mu$$

Thus the energy-momentum tensor is given by:

$$\boxed{T^{\mu\nu} = i g^{\mu\nu} \bar{\Psi} \gamma^\mu \partial_\mu \Psi - m g^{\mu\nu} \bar{\Psi} \Psi - i \bar{\Psi} \gamma^\mu \partial^\nu \Psi}$$

Finally, we need $B^{\mu\nu\rho}$, which is given by:

$$\boxed{B^{\mu\nu\rho} = \bar{\Psi}^A \gamma^\mu (S^{\mu\nu})_A^B \Psi_B}$$

where $S^{\mu\nu}$ is given by:

$$\boxed{S^{\mu\nu} = \begin{pmatrix} +(S_L^{\mu\nu})_a^c & 0 \\ 0 & -(S_R^{\mu\nu})_{\dot{c}}^{\dot{a}} \end{pmatrix}}$$

Combining our boxed results into equation (36.4.4) gives us the explicit form for the Belinfante tensor. While it is normally worth writing these results explicitly, doing so in this case promises to yield a horrible equation which will yield very little insight. As such, it is perhaps better to leave our result in this form.

Srednicki 36.5. Symmetries of fermion fields. Consider a theory with N massless Weyl fields ψ_j

$$\mathcal{L} = i\psi_j^\dagger \sigma^\mu \partial_\mu \psi_j$$

where the repeated index j is summed. This Lagrangian is clearly invariant under the $U(N)$ transformation,

$$\psi_j \rightarrow U_{jk} \psi_k$$

where U is a unitary matrix. State the invariance group for the following cases:

(a) N Weyl fields with a common mass M .

$$\mathcal{L} = i\psi^\dagger \sigma^\mu \partial_\mu \psi - \frac{m}{2} \psi \psi - \frac{m}{2} \psi^\dagger \psi^\dagger$$

Let's see if it's still true for $U(N)$:

$$\mathcal{L} \rightarrow i(U_{ij}\psi_j)^\dagger \sigma^\mu \partial_\mu (U_{ik}\psi_k) - \frac{m}{2} (U_{ij}\psi_j)(U_{ik}\psi_k) - \frac{m}{2} (U_{ij}\psi_j)^\dagger (U_{ik}\psi_k)^\dagger$$

This becomes:

$$\mathcal{L} \rightarrow i\psi^\dagger U^\dagger \sigma^\mu \partial_\mu U \psi - \frac{m}{2} \psi_j U_{ji}^T U_{ik} \psi_k - \frac{m}{2} \psi_j^\dagger U_{ji}^* U_{ik}^\dagger \psi_k^\dagger$$

Now we need to kill the U s. In the first term, as Srednicki says, the terms vanish if $U^\dagger U = 1$. By the way, don't be concerned about commutation with the Pauli matrix: the Pauli matrix is multiplied by the derivative; the result is a ket of derivatives, which is an *operator*. This will commute with the constant unitary matrix. We'll soon develop a new notation to make this point cleaner.

In the second and third terms, we need $U^T U = I$ in order to kill the U s; this implies that $U^T = U^{-1}$, which by our first condition implies that $U^T = U^\dagger$. This implies that $U = U^*$, which means that U must be real.

A real unitary matrix is called an orthogonal matrix, and so the group is invariant under $O(N)$.

(b) N massless Majorana fields

$$\mathcal{L} = \frac{i}{2} \Psi_j^T \mathcal{C} \gamma^\mu \partial_\mu \Psi_j$$

We can break the Majorana field into components of its Weyl fields. A Majorana field consists of only one Weyl field (the left-handed one on top, and the conjugate of the right-handed

one on the bottom). Expanding the expression in the Lagrangian, we then get only N unique terms, the same as when we dealt directly with Weyl fields (though there is now a factor of two, since the Weyl field appears twice per Majorana Field. The Lagrangian is then substantially unchanged from that of the first term of part (a), and so is the symmetry group. $U(N)$.

(c) N Majorana fields with a common mass m

$$\mathcal{L} = \frac{i}{2} \Psi_j^T \mathcal{C} \gamma^\mu \partial_\mu \Psi_j - \frac{1}{2} m \Psi_j^T \mathcal{C} \Psi_j$$

Same as part (b), except now the entire Lagrangian of part (a) is applicable, not merely the first term. The result is the same. $O(N)$.

(d) N massless Dirac fields

$$\mathcal{L} = i \bar{\Psi}_j \gamma^\mu \partial_\mu \Psi_j$$

Same as part (b), except now there are two $2N$ mass terms in the Lagrangian rather than N , since each Dirac field consists of two Weyl spinors. All of these terms can “mix” together. Hence, the group is enlarged to $U(2N)$.

(e) N Dirac fields with a common mass m

$$\mathcal{L} = i \bar{\Psi}_j \gamma^\mu \partial_\mu \Psi_j - m \bar{\Psi}_j \Psi_j$$

Same as part (c), except now there are twice as many terms, since each Dirac field consists of two Weyl spinors. $O(2N)$.