Srednicki Chapter 35

QFT Problems & Solutions

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Srednicki 35.1. Verify that equation 35.20 follows from 35.2 and 35.19.

This is simply a matter of combining 35.2 and 35.19, then manipulating the indices. We'll start with $\mu = 0$.

$$
\begin{aligned}
\bar{\sigma}^{0\dot{a}a} &= \varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}\delta_{b\dot{b}} \\
&= \varepsilon^{ab}\varepsilon^{\dot{a}b} \\
&= \begin{cases} 1 & \text{if } a = \dot{a} \\ 0 & \text{otherwise} \end{cases} \\
&= \delta_{a\dot{a}} \\
&= I
\end{aligned}
$$

where the third equality follows because when b is summed, the component with $b = a = \dot{a}$ vanishes, and the component where $b \neq a = \dot{a}$ remains, giving $1^2 = 1$ or $(-1)^2 = 1$.

Next for $\mu = i$ (that is to say, $\mu \neq 0$):

$$
\begin{aligned}
\bar{\sigma}^{i\dot{a}a} &= \varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}\sigma^{i}_{b\dot{b}} \\
&= \varepsilon^{ab}\sigma^{i}_{b\dot{b}}\varepsilon^{\dot{a}\dot{b}} \\
&= -\varepsilon^{ab}\sigma^{i}_{b\dot{b}}\varepsilon^{\dot{b}\dot{a}} \\
&= \sigma_2^{ab}\sigma^{i}_{b\dot{b}}\sigma_2^{\dot{b}\dot{a}} \\
&= (\sigma_2\sigma^{i}\sigma_2)^{a\dot{a}} \\
&= \left[(\sigma_2\sigma_i\sigma_2)^T\right]^{\dot{a}a} \\
&= \begin{cases} (-\sigma_1^T)^{\dot{a}a} & \text{if } i = 1 \\ (\sigma_2^T)^{\dot{a}a} & \text{if } i = 2 \\ (-\sigma_3^T)^{\dot{a}a} & \text{if } i = 3 \end{cases} \\
&= -\sigma_i^{\dot{a}a} \\
&= -\sigma^{i\dot{a}a}
\end{aligned}
$$

In the sixth and ninth equalities, note that $\sigma^i = \sigma_i$, since $i \neq 0$.

Combining these two results gives equation 35.20.

Srednicki 35.2. Verify that equation 35.21 is consistent with equations 34.9 and 34.10.

Equation 35.21 is:

$$(S_L)^{\mu\nu}_L = \frac{i}{4}\left(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\right)$$

Now we simply take $\mu = i$ and $\nu = j$ to symbolize that we're considering spatial indices only. The notation here is really confusing because the object $\sigma^{\mu\dot{a}a}$ looks exactly like the Pauli Matrices, σ. Here, we mean the invariant object:

$$(S_L)^{ij}_L = \frac{i}{4}\left(\sigma^i\bar{\sigma}^j - \sigma^j\bar{\sigma}^i\right)$$

And now we'll switch to the Pauli Matrices (using 35.2 and 35.20):

$$(S_L)^{ij}_L = -\frac{i}{4}\left(\sigma^i\sigma^j - \sigma^j\sigma^i\right)$$

which is:

$$(S_L)^{ij}_L = -\frac{i}{4}[\sigma^i, \sigma^j]$$

This gives:

$$(S_L)^{ij}_L = \frac{1}{2}\varepsilon^{ijk}\sigma^k$$

which is consistent with equation 34.9. □

In the same way, we now go back to 35.21, this time allowing $\mu = k$ and $\nu = 0$. We have:

$$(S_L)^{k0} = \frac{i}{4}\left(\sigma^k\bar{\sigma}^0 - \sigma^0\bar{\sigma}^k\right)$$

Switching to Pauli Matrices:

$$(S_L)^{k0} = \frac{i}{4}\left(\sigma^k + \sigma^k\right)$$

This gives:

$$(S_L)^{k0} = \frac{i}{2}\sigma^k$$

which is consistent with equation 34.10. □

Note 1: If you're wondering where the indices went, we chose to drop them. This is equivalent to working with matrices in the abstract, rather than individual components.

Note 2: In future, we will consider $\sigma^{\mu\dot{a}a}$ to be a "four-dimensional Pauli Matrix," which elucidates the confusing notation commented upon above.

Srednicki 35.3. Verify that equation 35.22 is consistent with equation 34.17.

Equation 35.22 is:

$$(S^{\mu\nu}_R)^{\dot{a}}{}_{\dot{b}} = -\frac{i}{4}\left(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\right)^{\dot{a}}{}_{\dot{b}}$$

We drop the indices.

$$(S_R^{\mu\nu}) = -\frac{i}{4}\left(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\right)$$

which gives:

$$(S_R^{\mu\nu}) = -\left[-\frac{i}{4}\left(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\right)^\dagger\right]^\dagger$$

and:

$$(S_R^{\mu\nu}) = -\left[-\frac{i}{4}\left(\sigma^{\dagger\nu}\bar{\sigma}^{\dagger\mu} - \sigma^{\dagger\mu}\bar{\sigma}^{\dagger\nu}\right)\right]^\dagger$$

The Pauli matrices are Hermitian, so:

$$(S_R^{\mu\nu}) = -\left[-\frac{i}{4}\left(\sigma^\nu\bar{\sigma}^\mu - \sigma^\mu\bar{\sigma}^\nu\right)\right]^\dagger$$

which is

$$(S_R^{\mu\nu}) = -\left[\frac{i}{4}\left(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\right)\right]^\dagger$$

Using equation 35.21:

$$(S_R^{\mu\nu}) = -\left(S_L^{\mu\nu}\right)^\dagger$$

Now we can reinsert the indices: at that point, the dagger will be acting on one complex number (albeit one complex number whose value shifts depending on the indices). It is not practical to take the transpose of one complex number, so we reduce the Hermitian conjugate to a complex conjugate. The result is equation 34.17.

Srednicki 35.4. Verify equation 35.5.

Applying equation 35.19:

$$\varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}\sigma^\mu_{a\dot{a}}\sigma^\nu_{b\dot{b}} = \sigma^\mu_{a\dot{a}}\bar{\sigma}^{\nu\dot{a}a}$$

Notice what this is: the two matrices are multiplied (inside indices) and the resultant matrix has its trace taken (outside indices). Dropping the index notation, we have:

$$\varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}\sigma^\mu_{a\dot{a}}\sigma^\nu_{b\dot{b}} = Tr(\sigma^\mu\bar{\sigma}^\nu)$$

Now we can test various cases. If both indices are 0, we take the trace of the (2x2) identity, which is 2. If both are the same nonzero index, then we get a negative sign (from the $\bar{\sigma}$) and the trace of a Pauli matrix squared, which is negative the trace of the identity, which is -2. If one index is 0 and the other is nonzero, then we have the trace of the identity times a Pauli matrix (possibly with a minus sign), and a Pauli matrix has trace zero. If the indices are nonzero and nonequal, we have the trace of the product of two Pauli matrices (possibly with a minus sign), which is the trace of the third Pauli matrix (possibly with a minus sign), and the Pauli matrix has trace 0. Combining these results, we have:

$$\varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}\sigma^\mu_{a\dot{a}}\sigma^\nu_{b\dot{b}} = \mathrm{diag}(2,-2,-2,-2)$$

which is

$$\varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}\sigma^\mu_{a\dot{a}}\sigma^\nu_{b\dot{b}} = -2g^{\mu\nu}$$

as expected.