Srednicki Chapter 34

QFT Problems & Solutions

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Srednicki 34.1. Verify that equation 34.6 follows from equation 34.1.

We take $\Lambda = 1 + \delta\omega$:

$$U(1+\delta\omega)^{-1}\psi U(1+\delta\omega) = L(1+\delta\omega)\psi([1+\delta\omega]^{-1}x)$$

Next we use equation 34.3, and $U(1+\delta\omega) = I + \frac{i}{2}\delta\omega M$:

$$(I - \frac{i}{2}\delta\omega M)\psi(I + \frac{i}{2}\delta\omega M) = (I + \frac{i}{2}\delta\omega S_L)\psi(x - x\delta\omega)$$

We do the multiplication on the left side, dropping the term with two differentials. On the right, we distribute:

$$\psi + \frac{i}{2}\psi\delta\omega M - \frac{i}{2}\delta\omega M\psi = I\psi(x - x\delta\omega) + \frac{i}{2}\delta\omega S_L\psi(x - x\delta\omega)$$

On the left, we rewrite using index notation. On the right, we expand in a Taylor Series: $\psi(x - x\delta\omega) = \psi(x) - x\delta\omega\partial^\mu\psi(x\delta\omega)$:

$$\psi + \frac{i}{2}\delta\omega_{\mu\nu}[\psi, M^{\mu\nu}] = \psi(x) - x\delta\omega\partial^\mu\psi(x) + \frac{i}{2}\delta\omega_{\mu\nu}(S_L^{\mu\nu})_a{}^b\psi_b(x - x\delta\omega)$$

The ψ terms cancel. Further, the last term already has a differential, so we keep only the leading term in the expansion of $\psi(x - x\delta\omega)$:

$$\frac{i}{2}\delta\omega_{\mu\nu}[\psi, M^{\mu\nu}] = -x\delta\omega\partial^\mu\psi(x) + \frac{i}{2}\delta\omega_{\mu\nu}(S_L^{\mu\nu})_a{}^b\psi_b(x)$$

Here comes the subtelty. Each of these terms is multiplied by $\delta\omega_{\mu\nu}$. This is an arbitrary antisymmetric tensor. Any symmetric coeffcients are irrelevent, since those terms must be equal to zero. Anti-symmetric coefficients result in nonzero terms, and so can be equated. In this equation, M and S are already antisymmetric. All that remains is to antisymmetrize the first term on the right side. We do this by writing the term twice:

$$\frac{i}{2}\delta\omega_{\mu\nu}[\psi, M^{\mu\nu}] = -\frac{1}{2}x^\nu\delta\omega_{\mu\nu}\partial^\mu\psi(x) - \frac{1}{2}x^\nu\delta\omega_{\mu\nu}\partial^\mu\psi(x) + \frac{i}{2}\delta\omega_{\mu\nu}(S_L^{\mu\nu})_a{}^b\psi_b(x)$$

Now switch the dummy indices in the second term:

$$\frac{i}{2}\delta\omega_{\mu\nu}[\psi, M^{\mu\nu}] = -\frac{1}{2}x^\nu \delta\omega_{\mu\nu}\partial^\mu\psi(x) - \frac{1}{2}x^\mu \delta\omega_{\nu\mu}\partial^\nu\psi(x) + \frac{i}{2}\delta\omega_{\mu\nu}(S_L^{\mu\nu})_a{}^b\psi_b(x)$$

Now we switch them back in the $\delta\omega$. This costs a minus sign, due to anti-symmetry:

$$\frac{i}{2}\delta\omega_{\mu\nu}[\psi, M^{\mu\nu}] = -\frac{1}{2}x^\nu \delta\omega_{\mu\nu}\partial^\mu\psi(x) + \frac{1}{2}x^\mu \delta\omega_{\mu\nu}\partial^\nu\psi(x) + \frac{i}{2}\delta\omega_{\mu\nu}(S_L^{\mu\nu})_a{}^b\psi_b(x)$$

This is:

$$\frac{i}{2}\delta\omega_{\mu\nu}[\psi, M^{\mu\nu}] = \left[-\frac{1}{2}x^\nu \partial^\mu\psi(x) + \frac{1}{2}x^\mu \partial^\nu\psi(x)\right]\delta\omega_{\mu\nu} + \frac{i}{2}\delta\omega_{\mu\nu}(S_L^{\mu\nu})_a{}^b\psi_b(x)$$

Now all the coefficients of $\delta\omega$ are antisymmetric, so we can equate them:

$$\frac{i}{2}[\psi, M^{\mu\nu}] = -\frac{1}{2}x^\nu \partial^\mu\psi(x) + \frac{1}{2}x^\mu \partial^\nu\psi(x) + \frac{i}{2}(S_L^{\mu\nu})_a{}^b\psi_b(x)$$

Finally, we multiply by $\frac{2}{i}$:

$$[\psi, M^{\mu\nu}] = -\frac{1}{i}\left(x^\nu\partial^\mu - x^\mu\partial^\nu\right)\psi(x) + (S_L^{\mu\nu})_a{}^b\psi_b(x)$$

Using the definition of $\mathcal{L}^{\mu\nu}$, and inserting the remaining Latin indices for consistency:

$$[\psi_a, M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\psi_a(x) + (S_L^{\mu\nu})_a{}^b\psi_b(x)$$

which is equation 34.6.

Srednicki 34.2. Verify that equations 34.9 and 34.10 obey equation 34.4.

Using equation 34.9:

$$[S_L^{ij}, S_L^{mn}] = \frac{1}{4}\varepsilon^{ijk}\varepsilon^{mno}[\sigma_k, \sigma_o]$$

Let k be either m or n (we can neglect the case where k is o, since the commutator will vanish). Then:

$$[S_L^{ij}, S_L^{mn}] = \frac{1}{4}\left(\varepsilon^{ijm}\varepsilon^{mno}[\sigma_m, \sigma_o] + \varepsilon^{ijn}\varepsilon^{mno}[\sigma_n, \sigma_o]\right)$$

Similarly, we let o be either i or j:

$$[S_L^{ij}, S_L^{mn}] = \frac{1}{4}\left(\varepsilon^{ijm}\varepsilon^{mni}[\sigma_m, \sigma_i] + \varepsilon^{ijn}\varepsilon^{mni}[\sigma_n, \sigma_i] + \varepsilon^{ijm}\varepsilon^{mnj}[\sigma_m, \sigma_j] + \varepsilon^{ijn}\varepsilon^{mnj}[\sigma_n, \sigma_j]\right)$$

Playing with the indices:

$$[S_L^{ij}, S_L^{mn}] = \frac{1}{4}\left(-\varepsilon^{ijm}\varepsilon^{inm}[\sigma_m, \sigma_i] + \varepsilon^{ijn}\varepsilon^{imn}[\sigma_n, \sigma_i] + \varepsilon^{ijm}\varepsilon^{njm}[\sigma_m, \sigma_j] - \varepsilon^{ijn}\varepsilon^{mjn}[\sigma_n, \sigma_j]\right)$$

Now this gets a little bit subtle. Note that we cannot use the summed properties of the Levi-Cevita symbol (such as $\varepsilon^{abc}\varepsilon^{dbc} = 2\delta^{ad}$), because these indices are not being summed over. The indices of the Levi-Cevita symbol are set once and for all by the choices on the left hand side. Instead, consider the first term. There are only three spatial dimensions, so i, j, and m must be 1, 2, or 3. Duplicates are not allowed by the definition of the Levi-Cevita symbol. It follows then that j must equal n. Since the two symbols are identical, they can only yield a positive one (or zero). Notice, therefore, that all the information of the Levi-Cevita symbols is contained in this statement: the term is nonzero only if (1) $j = n$, (2) $i \neq j$, and (3) $i \neq m$. We can repeat this analysis for all terms: the indices in conditions (1) and (3) will change, condition (2) will remain the same.

Note that (3) is redundant: this information is already encoded in the commutator. As for (1), we can more easily state that in a δ. Let's try to do so:

$$[S_L^{ij}, S_L^{mn}] = \frac{1}{4}\left(-\delta^{jn}[\sigma_m, \sigma_i] + \delta^{jm}[\sigma_n, \sigma_i] + \delta^{in}[\sigma_m, \sigma_j] - \delta^{im}[\sigma_n, \sigma_j]\right)$$

How do we impose the remaining condition, that $i \neq j$? It's already taken care of: if $i = j$, the first and third terms cancel, as do the third and fourth. Hence, we get a nonzero result if and only if $i \neq j$, just as required.

Next, recall that $[\sigma_a, \sigma_b] = 2i\varepsilon^{abc}\sigma_c = 4iS_L^{ab}$. So:

$$[S_L^{ij}, S_L^{mn}] = i\left(-\delta^{jn}S_L^{mi} + \delta^{jm}S_L^{ni} + \delta^{in}S_L^{mj} - \delta^{im}S_L^{nj}\right)$$

Finally, recall that for spatial (Latin) indices, $g = \delta$. Also, the S_L matrices are antisymmetric. Thus:

$$[S_L^{ij}, S_L^{mn}] = i\left(g^{im}S_L^{nj} - g^{jm}S_L^{in} - g^{in}S_L^{jm} + g^{jn}S_L^{im}\right)$$

which is equation 34.4 (for spatial indices). $\square$

Next we use equation 34.10:

$$[S^{k0}, S^{\ell 0}] = -\frac{1}{4}[\sigma_k, \sigma_\ell]$$

$$[S^{k0}, S^{\ell 0}] = -\frac{i}{2}\varepsilon^{k\ell m}\sigma_m$$

$$[S^{k0}, S^{\ell 0}] = -iS_L^{k\ell}$$

The answer from 34.4 is:

$$[S^{k0}, S^{\ell 0}] = i\left(g^{k\ell}S_L^{00} - g^{0\ell}S_L^{k0} - g^{k0}S_L^{0\ell} + g^{00}S_L^{k\ell}\right)$$

The first term vanishes since S_L is antisymmetric. The middle terms vanish because g is diagonal (recall that Latin indices represent spatial indices). $g^{00} = -1$. Hence,

$$[S^{k0}, S^{\ell 0}] = -iS_L^{k\ell}$$

So those match up. $\square$

Finally, we must consider the case:

$$\begin{aligned}
[S^{ij}, S^{k0}] &= \tfrac{i}{4}\varepsilon^{ij\ell}[\sigma_\ell, \sigma_k] \\
&= -\tfrac{1}{2}\varepsilon^{ij\ell}\varepsilon^{\ell km}\sigma_m \\
&= -\tfrac{1}{2}\varepsilon^{ij\ell}\varepsilon^{\ell ki}\sigma_i - \tfrac{1}{2}\varepsilon^{ij\ell}\varepsilon^{\ell kj}\sigma_j \\
&= \tfrac{1}{2}\varepsilon^{ij\ell}\varepsilon^{ik\ell}\sigma_i - \tfrac{1}{2}\varepsilon^{ij\ell}\varepsilon^{kj\ell}\sigma_j
\end{aligned}$$

As before, we replace the two ε functions with logical requirements, and find that these can be better implemented with delta functions. Specifically:

$$\begin{aligned}
[S^{ij}, S^{k0}] &= \tfrac{1}{2}\left(\delta^{jk}\sigma_i - \delta^{ik}\sigma_j\right) \\
&= i\left(g^{ik}S_L^{j0} - g^{jk}S_L^{i0}\right)
\end{aligned}$$

Is this what we would expect? Recall that i, j, k represent spatial indices only; g is diagonal, so any terms involving g^{i0}, g^{j0}, or g^{k0} must vanish. Equation 34.4 therefore gives exactly this. □

Srednicki 34.3. Show that the Levi-Cevita symbol obeys:

$$\boldsymbol{\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\gamma\sigma} = -\delta^\mu{}_\alpha\delta^\nu{}_\beta\delta^\rho{}_\gamma - \delta^\mu{}_\beta\delta^\nu{}_\gamma\delta^\rho{}_\alpha - \delta^\mu{}_\gamma\delta^\nu{}_\alpha\delta^\rho{}_\beta + \delta^\mu{}_\beta\delta^\nu{}_\alpha\delta^\rho{}_\gamma + \delta^\mu{}_\alpha\delta^\nu{}_\gamma\delta^\rho{}_\beta + \delta^\mu{}_\gamma\delta^\nu{}_\beta\delta^\rho{}_\alpha}$$

$$\boldsymbol{\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\rho\sigma} = -2(\delta^\mu{}_\alpha\delta^\nu{}_\beta - \delta^\mu{}_\beta\delta^\nu{}_\alpha)}$$

$$\boldsymbol{\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\nu\rho\sigma} = -6\delta^\mu{}_\alpha}$$

For the first: if $\mu\nu\rho$ is an even permutation of $\alpha\beta\gamma$, then we will get -1 (superscripts are inverse of subscripts). An odd permutation will give +1. This is exactly what the first line shows.

For the second: the same logic holds, except this time ρ and σ can be exchanged, giving a factor of 2.

For the third: the same logic holds, except this time ν, ρ, and σ can be exchanged, giving a factor of 6.

Srednicki 34.4. Consider a field $\boldsymbol{C^{a\ldots c\;\dot{a}\ldots\dot{c}}(x)}$, with N undotted indices and M dotted indices, that is furthermore symmetric on exchange of any pair of undotted indices, and also symmetric on exchange of any pair of dotted indices. Show that this field corresponds to a single irreducible representation (2n+1, 2n'+1) of the Lorentz Group, and identify n and n'.

Let's start by recognizing what this is. We're being asked to deal with the following:

$$(2,1)\otimes(2,1)\otimes\ldots\otimes(1,2)\otimes(1,2)\otimes\ldots$$

This is, of course, equal to:

$$(2\otimes 2\otimes\ldots, 2\otimes 2\otimes\ldots)$$

where there are N terms in the first part and M in the second. Next we need to determine the number of irreducible representations of this, and the dimensionality thereof. To determine the irreducible represntations, we use Young Tableaux (for a more complete introduction, see Sakurai section 6.5). In a young tableaux, each box represents one index, which in SU(2) can be 0 or 1. Boxes are then combined: a horizontal string of boxes represents a symmetric state; a vertical string of boxes represents an anti-symmetric state; boxes in any other arrangement represent a mixed symmetry.

To determine $2 \otimes 2$, then, we have:

$$\begin{array}{|c|}\hline \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \\ \hline\end{array} = \begin{array}{|c|c|}\hline & \\ \hline\end{array} \oplus \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$$

The first of these is symmetric and the second is antisymmetric. What about the dimensionality? The rule is that rows must be nondecreasing; the columns must increase. So the symmetric diagram is a triplet, it can be $\begin{array}{|c|c|}\hline 0 & 0 \\ \hline\end{array}$, $\begin{array}{|c|c|}\hline 0 & 1 \\ \hline\end{array}$, or $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}$. The antisymmetric diagram can only be $\begin{array}{|c|}\hline 0 \\ \hline 1 \\ \hline\end{array}$. Thus, $2 \otimes 2 = 3_S \oplus 1_A$.

Similarly, let's determine $2 \otimes 2 \otimes 2$. This is:

$$2 \otimes 2 \otimes 2 = \left(\begin{array}{|c|c|}\hline & \\ \hline\end{array} \oplus \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} \right) \otimes \begin{array}{|c|}\hline \\ \hline\end{array}$$

$$2 \otimes 2 \otimes 2 = \left(\begin{array}{|c|c|}\hline & \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \\ \hline\end{array} \right) \oplus \left(\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \\ \hline\end{array} \right)$$

$$2 \otimes 2 \otimes 2 = \left(\begin{array}{|c|c|c|}\hline & & \\ \hline\end{array} \oplus \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array} \right) \oplus \left(\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array} \right)$$

Notice that $\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}$ is not a legal diagram is SU(2), per the rules above. What about the dimensionality? The first can be $\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline\end{array}$, $\begin{array}{|c|c|c|}\hline 0 & 0 & 1 \\ \hline\end{array}$, $\begin{array}{|c|c|c|}\hline 0 & 1 & 1 \\ \hline\end{array}$, or $\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline\end{array}$. The others can be $\begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 \\ \cline{1-1}\end{array}$ or $\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$. Thus, $2 \otimes 2 \otimes 2 = 4_S + 2_M + 2_M$.

In the case at hand, we are absolutely required to consider only the symmetric case. The symmetric case will have N boxes in a horizontal row. As in the examples above, this horizontal row will have dimensionality N+1, since there are exactly N+1 ways to build the box (with no 1s, with one 1, ..., with all 1s).

Hence, the dimensionality is $(N + 1, M + 1)$. It is obvious that $N = 2n$ and $M = 2n'$.

It remains to show that this representation is irreducible. But that's the whole point of Young Tableaux: when all the $\otimes$ symbols are replaced by $\oplus$, the remaining Tableaux are *a priori* irreducible.